5.31 Convergence of Taylor Series

Let

$$f(x) = \sum_{j=0}^{\infty} a_j x^j \tag{1}$$

be a power series with radius of convergence $R > 0$ (as usual, R may be $+\infty$). By the term-by-term differentiation Theorem, $f'(x)$ exists for all $|x| < R$ and

$$f'(x) = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

By the same theorem, we know that the above series is differentiable on the interval $(-R, R)$ and

$$f''(x) = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2}$$

Continuing, we obtain

$$f^{(n)}(x) = \sum_{j=2}^{\infty} j(j-1)\cdots(j-n+1) a_j x^{j-n}$$

Notice that

$$f^{(n)}(0) = n(n-1)\cdots(n-n+1)a_n = n!a_n$$

so that (1) may be rewritten as

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j, \quad x \in (-R, R) \tag{2}$$

1. When does a Taylor series converge to its generating function?
2. How accurately can a function be approximated by its Taylor polynomials?

Taylor's Theorem

The following theorem is a generalization of the Mean Value Theorem.

Theorem 1. Taylor's Theorem
If f and its first derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval $[a, x]$ and $f^{(n)}$ is differentiable on the open interval (a, x), then there is a number $c \in (a, x)$ such that

(3)
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$
$$\cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Remark. Compare (3) with the text. Note that equation (3) remains unchanged if the interval $[a, x]$ is replaced by the interval $[x, a]$.

Definition. Taylor's Formula
If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each $x \in I$,

(4)
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where the remainder is given by the formula

(5)
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

In other words, Taylor's theorem says that for each $x \in I$,

$$f(x) = P_n(x) + R_n(x)$$

Now if $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at a converges to f on I and

(6)
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Theorem 2. Taylor Series for e^x

(7)
$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Proof. We need to prove that the remainder, $R_n(x) \to 0$ as $n \to \infty$ for all x. For $x = 0$ there is nothing to prove.
Suppose $x > 0$. Since the exponential is an increasing function, $1 < e^c \leq e^x$ for any $c \in [0, x]$. Hence

$$\begin{aligned} R_n(x) &= \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-0)^{n+1} \\ &= e^c\,\frac{x^{n+1}}{(n+1)!} \leq e^x\,\frac{x^{n+1}}{(n+1)!} \end{aligned}$$

Now the right-hand side approaches zero as $n \to 0$. Similarly for $x < 0$. This establishes (7). □

By an argument similar to the one given above we also have

(8)
$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n\,x^{2n+1}}{(2n+1)!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+1}}{(2n+1)!} \end{aligned}$$

so that

(9)
$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \end{aligned}$$

which follows either by mimicking the textbook's proof of (8) or by appealing to the Term-by-Term Differentiation Theorem from section 10.7.

Proposition 3. **Consequences of Theorem 2.**
Let $x \in \mathbb{R}$. Then

(10) $$e^x \geq 1 + x \quad \text{and}$$

(11) $$e^{ix} = \cos x + i \sin x, \quad i = \sqrt{-1}$$

The latter equation is usually referred to as **Euler's Identity**.

Proof. We have equality in (10) if $x = 0$. For $x > 0$ we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots > 1 + x$$

since the omitted terms are all positive. The inequality is obvious for $x \leq -1$ since the right-hand side is negative in that case. If $x \in (-1, 0)$ then $0 < |x| < 1$ and

$$\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} = \frac{x^{2n}}{(2n)!} \underbrace{\left(1 - \frac{|x|}{2n+1}\right)}_{>0} > 0$$

Thus

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} + \cdots \\ &= 1 + x + \left(\frac{x^2}{2!} + \frac{x^3}{3!}\right) + \cdots + \left(\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}\right) + \cdots \\ &= 1 + x + \sum_{n=1}^{\infty} \left(\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}\right) \\ &> 1 + x \end{aligned}$$

since the parenthetical quantities are positive.
For Euler's Identity,

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &\quad + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$

□

Remark. Making the substitution $x = -\pi$ in (11) yields

$$e^{-i\pi} = -1 \quad \text{or} \quad e^{-i\pi} + 1 = 0$$

The last equation is often called one of the most beautiful formulas in all of mathematics.

Estimating the Remainder

Theorem 4. The Remainder Estimation Theorem
If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \leq M$ for all t between x and a, inclusive, then the Remainder term in Taylor's Theorem satisfies

(12) $$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$$

If this condition holds for every n (and the other conditions of Taylor's Theorem are satisfied, then the Taylor series converges to the generating function, f. In other words, (6) holds.

If the series happens to be alternating, we have the following

Theorem 5. The Alternating Series Estimation Theorem
If the alternating series $\sum(-1)^{n+1} a_n$ satisfies the three conditions from Leibniz's Theorem, then for $n \geq N$, the partial sum

$$s_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n$$

approximates the sum L of the series with an error whose absolute value is less the a_{n+1}, i.e., is less than the absolute value of the first unused term.

Remark. In fact, we can say more. Let $\varepsilon_n = L - s_n$. Then ε_n has the same sign as the first unused term, a_{n+1}. The proof is very similar to the argument used to prove that $e^x \geq 1 + x$ for $x \in (-1, 0)$.

Example 1. Let $f(x) = \sqrt{x}$. Use the Taylor polynomials of order 1 and 4 to estimate $\sqrt{3/2}$. How accurate are these estimates?
In section 10.8 we saw that the Taylor Expansion of order 5 about $x = 1$ was

$$1 + \frac{x-1}{2} - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \frac{7}{256}(x-1)^5 + \underbrace{O\left((x-1)^6\right)}_{\text{Error Term}} \tag{13}$$

Using Theorem 5 we see that

$$|R_1(3/2)| \leq \frac{1}{8} \cdot \frac{1}{4} = \frac{1}{32}$$

and

$$|R_4(3/2)| \leq \frac{7}{256} \cdot \frac{1}{32} = \frac{7}{8192}$$

It follows that

$$P_1(x) = 1 + \frac{x-1}{2}$$

and

$$P_4(x) = 1 + \frac{x-1}{2} - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

Hence

$$\begin{aligned}\sqrt{3/2} &\approx P_1(3/2) \\ &= 5/4\end{aligned}$$

and

$$\begin{aligned}\sqrt{3/2} &\approx P_4(3/2) \\ &= 1 + \frac{1}{2}\left(\frac{3}{2}-1\right) - \frac{1}{8}\left(\frac{3}{2}-1\right)^2 \\ &\qquad + \frac{1}{16}\left(\frac{3}{2}-1\right)^3 - \frac{5}{128}\left(\frac{3}{2}-1\right)^4 \\ &= 1.22412\end{aligned}$$

Now by the remarks following Theorem 5 we know that

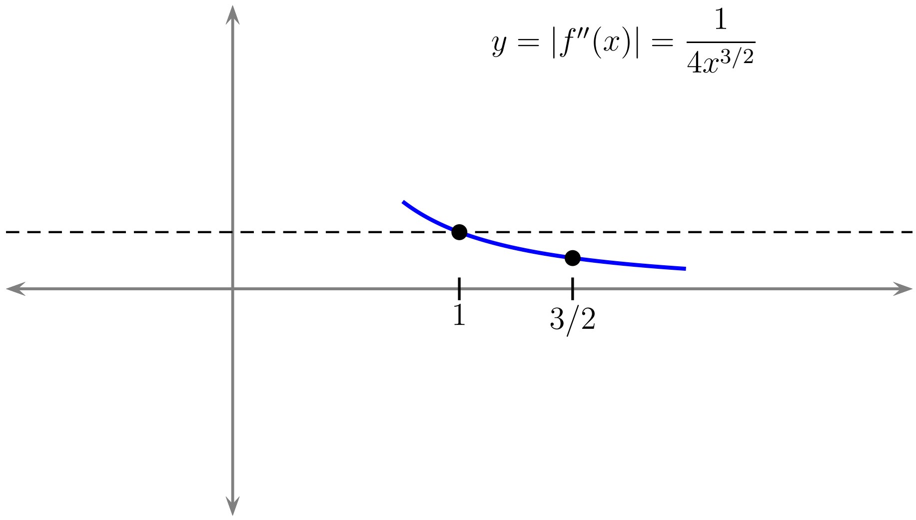
$$\begin{aligned}\varepsilon_4 &= \sqrt{3/2} - P_4(3/2) \\ &= \sqrt{3/2} - 1.22412\end{aligned}$$

is a positive number. It follows that the estimate

$$\sqrt{3/2} = 1.22412 + \varepsilon_4 > 1.22412$$

That is, our estimate is too low.

To see how good (or bad) these estimates are, we appeal to the Remainder Estimation Theorem.



It is clear that on the interval $I = [1, 3/2]$, $|f''(x)| \le M = |f''(1)| = 1/4$ since the second derivative is decreasing. It follows that

$$|R_1(3/2)| \le \frac{1}{4}\frac{|3/2-1|^{1+1}}{(1+1)!} = \frac{1}{32}$$

Similarly,

$$\left|f^{(5)}(x)\right| \le M = |f^{(5)}(1)| = \frac{105}{32}, \text{ on } I$$

so that

$$|R_4(3/2)| \le \frac{105}{32}\frac{|3/2-1|^{4+1}}{(4+1)!} = \frac{7}{8192}$$

In this case, the error estimates obtained using either Theorem 4 or 5 agree.

Now, for example, since

$$\sqrt{\frac{3}{2}} = P_4\left(\frac{3}{2}\right) + R_4\left(\frac{3}{2}\right)$$

we have

$$\begin{aligned}\left|\sqrt{\frac{3}{2}} - P_4\left(\frac{3}{2}\right)\right| &= \left|\sqrt{\frac{3}{2}} - 1.22412\right| \\ &= \left|R_4\left(3/2\right)\right| \\ &\le \frac{7}{8192} \approx 0.000854492\end{aligned}$$

or

$$\underbrace{1.22412 - 0.000854492}_{1.22327} \le \sqrt{\frac{3}{2}} \le \underbrace{1.22412 + 0.000854492}_{1.22498}$$