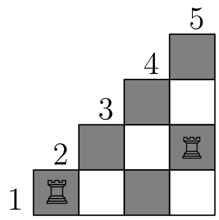
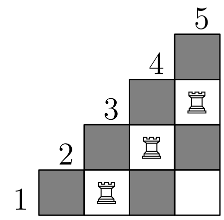


(a) $\sigma = 125/3/4$

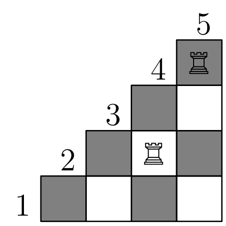


(b) $\delta = 135/24$

Figure 1: Nonattacking rooks and set partitions

1. (20 points) Consider the set partition $\sigma = 125/3/4 \in \left\{ {[5] \atop 3} \right\}$ in standard block notation. Figure 1a shows how to represent σ on a triangular chessboard using two nonattacking rooks. Figure 1b shows the representation for the set partition $\delta = 135/24$. *Note:* The colored lines are included for convenience.

   (a) Use the board below to represent the partition $\gamma = 1/245/3$ by placing a rook in one or more squares. *Note:* The lines in Figure 1 were for your convenience only and are normally not included on rook diagrams.



   (b) In general, how many (nonattacking) rooks and what size triangular board are needed to represent $\omega \in \left\{ {[n] \atop k} \right\}$?

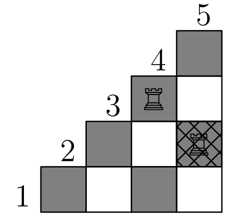
   Solution:

   We need an $(n-1) \times (n-1)$ triangular board with $n-k$ rooks.

   (c) It is a fact that there is a one-to-one correspondence between the number of nonattacking rook representations on a triangular chessboard and $B([n])$ (the number of set partitions on $[n]$). Now suppose that a rook is **forbidden** on the marked square in the figure below. Using exactly two nonattacking rooks, how many of the partitions in $B([5])$ can be represented under this restriction? Also, using block notation, give an example of a (two-rook) partition that is forbidden under this restriction.

   Solution:

   Placing rooks as shown would yield the partition 1/25/34. It follows that this is a forbidden partition. The quickest way is to enumerate all legal two-rook positions that include the indicated square and subtract the result from $\left\{ {5 \atop 3} \right\} = 25$. It turns out that there are 4 forbidden partitions, so the final answer is 21.



2. (20 points) A grocery store sells apples, oranges, and grapefruits. *Assume that each kind of fruit is identical and that the chosen order does not matter.*

   (a) If there is an unlimited supply of each kind, in how many ways can 15 fruits be chosen. EXPRESS YOUR ANSWER AS AN INTEGER.

   Solution:

$$\left(\!\!\binom{3}{15}\!\!\right) = \binom{3+15-1}{15} = \binom{17}{15} = 136$$

   (b) Suppose that there are an unlimited supply of apples but there only 14 oranges and 12 grapefruits in the store. In how many ways can 15 fruits be chosen. EXPRESS YOUR ANSWER AS AN INTEGER.

   Solution:

   The following collections are impossible.

Anything with...	**Count**
15 oranges	1
15 grapefruits	1
14 grapefruits	2
13 grapefruits	3

   It follows then that there are $136 - 7 = 129$ ways to select the fruit.

3. (10 points) Show that there exists a positive integer n so that $1 + 44 + 44^2 + \cdots + 44^n$ is divisible by 17. *Hint:* For example, the statement is false if 17 is replaced by 4 or 11.

Solution:

Let r_n be the remainder when $s_n = \sum_{i=0}^{n} 44^i$ is divided by 17. Then $\{r_n\}_{n=0}^{17} \subset \{0, 1, 2, \ldots, 16\}$ and by the Pigeonhole Principle, there must be indices $0 \le j < k \le 17$ such that $r_j = r_k$. If $r_j = 0$ then we are done. Otherwise, 17 divides $s_k - s_j = \sum_{i=j+1}^{k} 44^i = 44^{j+1} \sum_{i=0}^{k-j-1} 44^i$. Now since 17 is prime and since it does not divide 44, it must divide $\sum_{i=0}^{k-j-1} 44^i$ and we are done.

4. (10 points) Show that

$$\sum_{k \text{ even}} \binom{n}{k} 2^k = \frac{3^n + (-1)^n}{2}$$

Solution:

Notice that $\left\{\frac{(1+(-1)^k)}{2}\right\}_{k \ge 0} = \{1, 0, 1, 0, \ldots\}$.

$$\begin{aligned}
\sum_{k \text{ even}} \binom{n}{k} 2^k &= \sum_{k} \binom{n}{k} 2^k \, \frac{(1 + (-1)^k)}{2} \\
&= \frac{1}{2} \sum_{k} \binom{n}{k} 2^k + \frac{1}{2} \sum_{k} \binom{n}{k} (-2)^k \\
&= \frac{(1+2)^n + (1-2)^n}{2}
\end{aligned}$$

5. (10 points) Let $p \in \mathbb{P}$ and let $\left\{ {[n] \atop k} \right\}_p$ denote the collection of all set partitions with k blocks such that each block has no fewer than p elements. For example, $14/269/3578 \in \left\{ {[9] \atop 3} \right\}_2$ since each of the 3 blocks contains at least 2 elements. As usual, let $\left\{ {n \atop k} \right\}_p = \left| \left\{ {[n] \atop k} \right\}_p \right|$ and call such numbers *modified* Stirling set numbers.

   Prove that

   (1) $$\left\{ {n+1 \atop k} \right\}_p = k\left\{ {n \atop k} \right\}_p + \binom{n}{p-1}\left\{ {n-p+1 \atop k-1} \right\}_p, \quad n \geq p$$

   Solution:

   Question - In how many ways can we partition $[n+1]$ into k blocks under the given restriction?

   LHS - By definition.

   RHS - We use the distinguished element argument. There are two cases: either $n+1$ appears in a block of size p or it does not.

   Case 1: $n+1$ is in a block of size p. In that case, there are $\binom{n}{p-1}$ ways to choose the rest of the elements in the block, followed by $\left\{ {n-(p-1) \atop k-1} \right\}_p$ ways to partition the remaining elements into $k-1$ blocks with no block having fewer than p elements. So by the product rule, there are $\binom{n}{p-1}\left\{ {n-(p-1) \atop k-1} \right\}_p$ ways to partition $[n+1]$ in this case.

   Case 2: $n+1$ appears in a block of size greater than p. In that case, there are $\left\{ {n \atop k} \right\}_p$ ways to first partition $[n]$ into k blocks (under the given restriction) and then we may insert $n+1$ into any of the k blocks. So by the product rule, there are $k\left\{ {n \atop k} \right\}_p$ ways to do this.

   Since the two cases are disjoint, the result now follows by the sum rule.

6. (10 points) For $m \in \mathbb{N}$, show that

$$\sum_{k=0}^{m} \binom{n}{k}(-1)^k = (-1)^m \binom{n-1}{m}$$

for any integer n. For example, $\sum_{k=0}^{6} \binom{-2}{k}(-1)^k = (-1)^6 \binom{-2-1}{6} = 28$.

Hint: Use telescoping sums.

Solution:

Use telescoping sum together with the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Thus

$$\begin{aligned}
\sum_{k=0}^{m} \binom{n}{k}(-1)^k &= \sum_{k=0}^{m} \left\{ \binom{n-1}{k}(-1)^k + \binom{n-1}{k-1}(-1)^k \right\} \\
&= \sum_{k=0}^{m} \binom{n-1}{k}(-1)^k - \sum_{k=0}^{m} \binom{n-1}{k-1}(-1)^{k-1} \\
&= \sum_{k=0}^{m} \binom{n-1}{k}(-1)^k - \sum_{k=0}^{m-1} \binom{n-1}{k}(-1)^k \\
&= (-1)^m \binom{n-1}{m} + \sum_{k=0}^{m-1} \binom{n-1}{k}(-1)^k - \sum_{k=0}^{m-1} \binom{n-1}{k}(-1)^k \\
&= (-1)^m \binom{n-1}{m} + 0
\end{aligned}$$

as desired.

7. (20 points) This is a continuation of the modified Stirling set number problem from the in-class portion of the exam. We repeat it here for convenience.

   Let $p \in \mathbb{P}$ and let $\left\{ {[n] \atop k} \right\}_p$ denote the collection of all set partitions with k blocks such that each block has no fewer than p elements. Let $\left\{ {n \atop k} \right\}_p = \left| \left\{ {[n] \atop k} \right\}_p \right|$.

   We have

$$\left\{ {n+1 \atop k} \right\}_p = k \left\{ {n \atop k} \right\}_p + \binom{n}{p-1} \left\{ {n-p+1 \atop k-1} \right\}_p, \quad n \geq p \tag{2}$$

   (a) Let $B([n])_2$ denote the collection of all set partitions on $[n]$ with block size no smaller than 2 and let $b(n)_2 = |B([n])_2| = \sum_k \left\{ {n+1 \atop k} \right\}_2$. For example, according to Table 1 below, $b(6)_2 = 1 + 25 + 15 = 41$. Find a general formula for $b(n)_2$ in terms of the Bell numbers $b(n)$. *Hint:* See the derivation of subfactorials (equation (5) here).

   Solution:

   Let $B([n])$ denote the collection of all set partitions on $[n]$, so that the Bell numbers are given by $b_n = |B([n])|$, and, for this problem, let's call $b(n)_2$ the modified Bell numbers. It turns out that there are two equivalent formulas.

   Formula 1: After filling in the missing entries in Table 1 below, it is easy to *guess* that

$$b_n = b(n+1)_2 + b(n)_2 \tag{3}$$

   It now follows by induction that

$$b(n)_2 = \sum_{k=0}^{n-1} b_{n-1-k}(-1)^k + (-1)^n, \ n \geq 1 \tag{4}$$

   We will prove (3) by finding a bijection between $B([n])$ and $B([n])_2 \cup B([n+1])_2$. So let $\omega \in B([n])$ and define $\Gamma\colon B([n]) \to B([n])_2 \cup B([n+1])_2$ by the following rule:

   > If ω contains no singleton blocks, then let $\Gamma(\omega) = \omega \in B([n])_2$. On the other hand, suppose that ω contains $k-j+1$ singleton blocks, say $\omega = B_1/B_2/\cdots/B_{j-1}/\{x_j\}/\{x_{j+1}\}/\cdots/\{x_k\}$. Notice that we have placed all singleton blocks at the end. Now let $\Gamma(\omega) = B_1/B_2/\cdots/B_{j-1}/\{x_j x_{j+1} \cdots x_k\,(n+1)\} \in B([n+1])_2$. That is, we combine all the singletons into a single block and add $n+1$ to this new block. Since the presence of $n+1$ identifies the block with the singleton elements from the pre-image, the map is clearly invertible and (3) is established.

   Now let's do a sanity check and let MMA generate the first 11 modified Bell numbers using (4). We get

$$1, 0, 1, 1, 4, 11, 41, 162, 715, 3425, 17722$$

   Notice that according to (4), $b(0)_2 = 1$.

Solution:

Formula 2: We claim that $b(n)_2$ can also be written as

$$b(n)_2 = \sum_k (-1)^{n-k} \binom{n}{k} b_k \tag{5}$$

Following the hint, let $B(n,k)$ count the number of set partitions with exactly k singletons. First notice that $B(n,0) = b(n)_2$. Now consider all partitions with exactly k singletons. There are $\binom{n}{k}$ ways to choose the singletons and $B(n-k,0)$ ways to partition the remaining elements under the given restriction. So by the product rule, $B(n,k) = \binom{n}{k} B(n-k,0)$. Now a partition has zero singletons, or one singleton, etc., so by the sum rule

$$\begin{aligned} b_n &= \sum_{k=0}^{n} B(n,k) \\ &= \sum_k \binom{n}{k} B(n-k,0) = \sum_k \binom{n}{n-k} B(n-k,0) \\ &= \sum_k \binom{n}{k} B(k,0) \end{aligned}$$

Now by inversion

$$b(n)_2 = B(n,0) = \sum_k (-1)^{n-k} \binom{n}{k} b_k$$

and (5) is established.

If we let MMA generate the first 11 modified Bell numbers using (5), we get

$$1, 0, 1, 1, 4, 11, 41, 162, 715, 3425, 17722$$

as we saw for the first formula. Notice that both are consistent with the hand-generated values in Table 1.

(b) Use (2) to fill in the 4 missing entries in row 8 of Table 1 below.

$n \backslash k$	1	2	3	4	5	6	$b(n)_2$
2	1						1
3	1						1
4	1	3					4
5	1	10					11
6	1	25	15				41
7	1	56	105				162
8	1	119	490	105			715
9	1	246	1918	1260			3425
10	1	501	6825	9450	945		11722

Table 1: Modified Stirling set numbers, $\left\{ {n \atop k} \right\}_p$