1. (20 points) Consider the sequence $\{a_n\}_{n\geq 0}$ defined by the recursion below and answer the questions that follow.

$$a_{n+3} = 2a_{n+2} - a_n, \quad a_0 = 1,\, a_1 = 3,\, a_2 = 4 \tag{1}$$

(a) Find the next 3 terms in this sequence.

Solution:

The first 12 terms are $1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322$.

(b) Find the closed form of the generating function $A(x) = \sum_n a_n x^n$.

Solution:

According to the Wilf Rules, (1) is equivalent to the equation

$$\frac{A(x) - 1 - 3x - 4x^2}{x^3} = 2\frac{A(x) - 1 - 3x}{x^2} + A(x)$$

Rearranging yields

$$A(x)(1 - 2x + x^3) = 1 + x - 2x^2$$

Thus

$$A(x) = \frac{1 + x - 2x^2}{1 - 2x + x^3}$$

2. (20 points) How many 3-digit positive integers are divisible by at least one of the numbers in the set $\{6, 7, 11\}$? For example, there are $\lfloor \frac{999}{6} \rfloor - \lfloor \frac{100}{6} \rfloor = 166 - 16 = 150$ 3-digit numbers that are divisible by 6. *Express your answer as a positive integer.*

Solution:

Throughout this solution an integer is a 3-digit positive integer. Let p_6 be the property that an integer is divisible by 6. Then the number of integers that are divisible by 6 is $N(p_6) = 150$. Similarly,

$$N(p_7) = \left\lfloor \frac{999}{7} \right\rfloor - \left\lfloor \frac{100}{7} \right\rfloor = 128$$

$$N(p_{11}) = \left\lfloor \frac{999}{11} \right\rfloor - \left\lfloor \frac{100}{11} \right\rfloor = 81$$

Similarly,

$$N(p_6 p_7) = 21$$

$$N(p_6 p_{11} = 14$$

$$N(p_7 p_{11}) = 11$$

Finally,

$$N(p_6 p_7 p_{11}) = 2$$

Now there are 900 3-digit integers, so by PIE, the number of integers that are not divisible by at least one of 6, 7, or 11 is

$$\begin{aligned} N_0 &= 900 - (150 + 128 + 81) + (21 + 14 + 11) - 2 \\ &= 585 \end{aligned}$$

It follows that there are 315 integers that are divisible by at least one of these numbers.

3. (10 points) Let $\pi = (\pi_1\ \pi_2\ \cdots\ \pi_n) \in S_n$, here $n > 1$. Recall that the pair $(\pi_j, \pi_k),\ 1 \le j < k \le n$ is called an inversion pair if $\pi_j > \pi_k$. Recall that a permutation is called even (odd) if it has an even (odd) number of inversions.

   Now let $\tau = (2\ 1\ 3\ \cdots\ n) \in S_n$. To be clear, $\tau(1) = 2$, $\tau(2) = 1$, and $\tau(k) = k$ for $3 \le k \le n$. (τ is called a transposition.) Show that $\pi\tau \in S_n$ has a different parity than $\pi \in S_n$. That is, if π is even then $\pi\tau$ is odd and vice-versa.

   Solution:

   Let π and τ be as described above and let $\delta = \pi\tau$. We claim that $\delta = (\pi_2\ \pi_1\ \pi_3\ \cdots\ \pi_n)$. Now if the claim is true, then either $E(\delta) = E(\pi) \setminus \{(\pi_1, \pi_2)\}$ or $E(\delta) = E(\pi) \cup \{(\pi_2, \pi_1)\}$. In other words, $|E(\delta)| = |E(\pi)| \pm 1$, as desired.

   To prove the claim, notice that $\delta(1) = \pi(\tau(1)) = \pi(2) = \pi_2$, $\delta(2) = \pi(\tau(2)) = \pi(1) = \pi_1$, and $\delta(j) = \pi(j) = \pi_j$ for $j \ge 3$.

4. (10 points) Let s_n count the number of ways to break an n-semester day into two parts with one holiday during the first part and two (indistinguishable) holidays during the second part. In class we used generating functions to show that

   $$s_n = \sum_k \binom{k}{1}\binom{n-k}{2} = \binom{n+1}{4} \qquad (2)$$

   Use a combinatorial proof to show that $s_n = \binom{n+1}{4}$. *Look to the board for a possible hint.*

   Solution:

   This is exercise 8.13 from our textbook. See the solution on page 200.

   Here's another proof based on the hint that I wrote on the board. The right-hand side of (2) counts the number of ways to choose a 4-subset from $[n+1]$. Given such a subset, say $1 \le a < b < c < d \le n+1$, we observe that $a \le b-1 < c-1 < d-1 \le n$ and assign the semester break to day $b-1$ and the three holidays to days a, $c-1$, and $d-1$. Notice that this procedure is reversible even if the first holiday falls on the last day of the first "half" of the semester, in which case $a = b-1$. However, this presents no difficulties.

5. (20 points) Let $p \in \mathbb{P}$ and let $\left[\substack{[n]\\k}\right]_p$ be the set of all $\pi \in S_n$ such that each of the k cycles contains at least p elements. For example, $(137)(26458) \in \left[\substack{[8]\\2}\right]_3$ since both cycles have at least 3 elements. On the other hand, $(15)(2436) \notin \left[\substack{[6]\\2}\right]_3$ since the first cycle has only 2 elements.

   (a) Now set $\left[\substack{0\\0}\right]_p = 1$ and for $n > 0$, let $\left[\substack{n\\k}\right]_p = \left|\left[\substack{[n]\\k}\right]_p\right|$. Prove that

   (3) $$\begin{bmatrix} n+1 \\ k \end{bmatrix}_p = n\begin{bmatrix} n \\ k \end{bmatrix}_p + \binom{n}{p-1}(p-1)!\begin{bmatrix} n-p+1 \\ k-1 \end{bmatrix}_p, \quad n \geq p$$

   Solution:

   We used the distinguished element argument. The left-hand side counts the number of permutations on $[n+1]$ with exactly k cycles such that no cycle has fewer than p elements.

   Now $n+1$ either appears in a cycle that contains more than p elements or it appears in a cycle with exactly p elements. In the first case, we can choose a permutation from $\left[\substack{[n]\\k}\right]$ and then we may place $n+1$ into any of the cycles, after each element. Now by the product rule there are $\left[\substack{n\\k}\right] \cdot n$ ways to do this.

   Otherwise, $n+1$ is in a cycle with exactly $p-1$ elements. So there are $\binom{n}{p-1}$ ways to choose that elements that are in the same cycle is $n+1$, $(p-1)!$ ways to arrange those elements within the cycle, followed by $\left[\substack{n+1-p\\k-1}\right]_p$ ways to arrange the remaining $k-1$ cycles. So by the product rule, there are $\binom{n}{p-1}(p-1)!\left[\substack{n-p+1\\k}\right]$ ways to create a permutation in this case.

   Since the two cases are clearly disjoint, the result follows by the sum rule.

   (b) Let $D([n])_2$ denote the set of all permutations such that each cycle has at least 2 elements. Now let $d_0 = 1$ and for $n > 0$, let $d_n = |D([n])_2|$. Find the next 5 terms in this sequence. That is, find d_1, d_2, d_3, d_4, d_5.

   Solution:

   Including d_0, these are just the first 6 derangement numbers, which are the sums across the first 6 rows of Table 3. That is, 1, 0, 1, 2, 9, 44.

6. (20 points) Let $\{f_n\}_n$ be the Fibonacci numbers. For $n \geq m$, give a **combinatorial** proof of the identity below.

$$f_{n+m} = \sum_{k=0}^{m} \binom{m}{k} f_{n-k} \tag{4}$$

Solution:

The left-hand side counts the number of ways to cover B_{n+m}. For the right-hand side, we condition on the number of dominos within the first m tiles. So label the first m tiles from 1 through m. If they are all squares, they cover a B_m board in only one way and there are f_n ways to cover the rest of the board. So by the product rule, there are $1 \cdot f_n = \binom{m}{0} f_n$ ways to cover the board in this case. Now suppose there is one domino within the first m tiles. Then these tiles cover B_{m+1} and there are $\binom{m}{1}$ ways to do this. Since there are f_{n-1} ways to cover the squares that remain, the product rule implies that there are $\binom{m}{1} f_{n-1}$ ways to cover B_{n+m} in this case.

In general, there are $\binom{m}{k}$ ways to arrange k dominos within the first m tiles (covering B_{m+k}) and f_{n-k} ways to cover the remaining squares. Once again we apply the product rule. Now since the first m tiles may contain zero dominos, or one domino, or two dominos, etc. and since these cases are disjoint, the result now follows by the sum rule.