1. Let $\{a_n\}_{n\geq 0}$ be given by the recursion below. Answer the questions that follow.

$$a_{n+2} = 3a_{n+1} + a_n + 1, \quad a_0 = 1, a_1 = 4$$

(a) Find the next 3 terms in the sequence.

Solution:

Here are the first 12 terms: $1, 4, 14, 47, 156, 516, 1705, 5632, 18602, 61439, 202920, 670200$

(b) Find the closed form of the generating function $A(x) = \sum_{n\geq 0} a_n x^n$.

Solution:

The recursion implies

$$\frac{A(x) - 1 - 4x}{x^2} = \frac{3(A(x) - 1)}{x} + A(x) + \frac{1}{1-x}$$

Rearranging and solving for $A(x)$ yields

$$A(x) = \frac{1}{(1-x)(1-3x-x^2)}$$

2. Recall that the English alphabet contains 21 consonants and the 5 vowels, *a,e,i,o,u*.

(a) If repeated letters are forbidden, how many 8-letter strings contain exactly 5 vowels?

Solution:

As we saw in class, this is $\binom{5}{5}\binom{21}{3}8! = 53625600$.

(b) If repeated letters are allowed, how many 8-letter strings contain exactly 5 vowels? For example, unlike part (a), the string *bacaituo* is now permitted.

Solution:

There are $\binom{8}{5}$ ways to choose the locations of the vowels, 5^5 ways to choose the vowels that occupy those locations, and finally, 21^3 ways to fill the remaining positions with consonants. So by the product rule, there are $\binom{8}{5}5^5 21^3 = 1620675000$ possible strings.

3. Let d_n be the n^{th} derangement number, that is, let $d_n = !n$. Give a combinatorial proof of the identity below.

$$d_{n+1} = nd_n + nd_{n-1},\ n \geq 0,\ d_{-1} = 0,\ d_0 = 1$$

Hint: Recall that if $\pi \in S_n$ is a derangement, then π has no singleton cycles.

Solution:

The left-hand side counts the number of derangements on $[n+1]$.

For the right-hand side, in any derangement on $[n+1]$, $n+1$ is in a two-cycle or not.

Case i. If $n+1$ is in a two-cycle, there $\binom{n}{1}$ ways to choose the element paired with $n+1$ and d_{n-1} to derange the remaining $n-1$ elements. So by the product rule, there are $n\,d_{n-1}$ ways to create a derangement with $n+1$ in a two-cycle.

Case ii. Otherwise, we may write any derangement of $[n]$ as a product of cycles (with no singletons), and there are n positions where we can insert $n+1$. So by the product rule, there are $n\,d_n$ derangements where $n+1$ is not in a two-cycle.

Since the cases are disjoint, the result now follows by the sum rule.

4. Find the number of nonnegative integer solutions to the equation below subject to the restrictions that follow.

(1) $$x_1 + x_2 + x_3 = 28,$$

where $0 \leq x_1 \leq 9$, $0 \leq x_2 \leq 8$, and $0 \leq x_3 \leq 17$. Recall that the number of *nonnegative* integer solutions to (1) is given by the multichoose coefficient $\left(\!\binom{3}{28}\!\right)$.

Solution:

Throughout this proof, solutions to (1) will always mean nonnegative integer solutions. So let p_1 be the condition that $x_1 \geq 10$ and let $N(p_1)$ count the number of solutions to the equation $x_1 + x_2 + x_3 = 28$ with the added restriction to x_1. It is easy to see that this is equivalent to the number of solutions to $x_1' + 10 + x_2 + x_3 = 28$ or $x_1' + x_2 + x_3 = 18$ so that $N(p_1) = \left(\!\binom{3}{18}\!\right)$. In a similar manner, let p_2 be the condition that $x_2 \geq 9$ and p_3 be the condition that $x_3 \geq 18$. Then $N(p_2) = \left(\!\binom{3}{19}\!\right)$ and $N(p_3) = \left(\!\binom{3}{10}\!\right)$. Easy calculations show that $N(p_1p_2) = \left(\!\binom{3}{9}\!\right)$, $N(p_1p_3) = \left(\!\binom{3}{0}\!\right) = 1$, and $N(p_2p_3) = \left(\!\binom{3}{1}\!\right)$. Finally, $N(p_1p_2p_3) = 0$.

Then the number of solutions the (1) under the initial restrictions is given by

$$\begin{aligned} N_0 &= \left(\!\binom{3}{28}\!\right) - \left(\left(\!\binom{3}{18}\!\right) + \left(\!\binom{3}{19}\!\right) + \left(\!\binom{3}{10}\!\right)\right) + \left(\left(\!\binom{3}{9}\!\right) + \left(\!\binom{3}{0}\!\right) + \left(\!\binom{3}{1}\!\right)\right) - 0 \\ &= \binom{30}{28} - \left(\binom{20}{18} + \binom{21}{19} + \binom{12}{10}\right) + \left(\binom{11}{9} + 1 + \binom{3}{1}\right) \\ &= 28 \end{aligned}$$

By the way, the fact that there are 28 solutions is just a coincidence.

5. Let $c_0 = 1$ and for $n > 0$, let c_n count the number of nonnegative integer sequences $t_1, t_2, \ldots, t_n$ satisfying $t_1 + t_2 + \cdots + t_j \geq j$ and $\sum_{k=1}^{n} t_k = n$. For example, $c_3 = 5$ since the only such sequences are

$$111 \quad 120 \quad 210 \quad 201 \quad 300$$

Show that $\{c_n\}_{n\geq 0}$ are the Catalan numbers.

Solution:

We claim that there is a bijection between the set of legal strings of pairs of parentheses in S_n and the set T_n of sequences described above.

So let s be string in S_n. As we read s from left to right, we create a sequence $t = t_1, t_2, \ldots, t_n \in T_n$, as follows.

> t_1 counts the number of left parentheses prior to the first right parenthesis and for $1 < k \leq n$, t_k counts the number of left parentheses between the $(k-1)^{\text{st}}$ right parenthesis and the k^{th} right parenthesis.

Notice that since s contains n pairs of parentheses, $\sum_{k=1}^{n} t_k = n$, and the condition that $t_1 + t_2 + \cdots + t_j \geq j$ is equivalent to fact that the number of right parentheses never exceeds the number of left parentheses as the string is read from left to right.

For example, let

$$s = \underbrace{(\,(}_{t_1=2}\,)^1 \underbrace{(\,(\,(}_{t_2=3}\,)^2\,)^3\ \underbrace{(}_{t_4=1}\,)^4\,)^5\,)^6 \in S_6.$$

Here we labeled the right parentheses 1 through 6. Notice that $t_3 = t_5 = t_6 = 0$, so that $t = 230100$. A careful inspection will confirm that $t \in T_6$. Evidently, this process is reversible.

For example, let $t = 2101 \in T_4$. Then $s = (\,(\,)\,(\,)\,)\,(\,) \in S_4$.

The result follows.