Date	**Section**	**Exercises**** (QC - Quick Check and CE - Class Exercises)
10/09	Inversions	Inversions - 2, 3, 4(a)
10/11	6.1	QC - 1, 2; CE - 26, 27
10/13	6.1	QC - 3; CE - 1, 2, 3, 5, 18, 28
10/16	6.1	CE - 30(b) - Here, distinct means labeled. , 33
10/27	7.1	QC - 1; CE - 17, 19-21
10/30	7.1	QC - 2, 3; CE - 9, 10, 36
11/01	7.2	QC - 1; CE - 22-25
11/06*	8.1	CE - 9; Let $b_n = n^2/3^n$ and $B(x) = \sum_{n\geq 0} b_n x^n = (1+x+x^2)^{-1}$. How should we refer to b_n within the context of this class? What about $B(x)$?
11/08*	8.1	See below.
11/10*	8.1	CE - 47; Also, see below.
11/10*	8.1	CE - 39, 40; Also, see below.
11/13*	8.1	See below.

10/11 Show the if $\pi^2 = \text{id}$ (the identity permutation), then all cycles have length 1 or 2.

10/11

1. Let $j, k \in \mathbb{P}$ with $j < k$ and let $\tau = (j, k)$ be a transposition. Find a formula for $i(\tau)$, the inversion number of τ. First let $\sigma = (26)$ and show that $\sigma = (56)(45)(34)(23)(34)(45)(56)$. Does this representation of σ offer any advantages?

2. Show that for $n \geq 2$,
$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k = 0$$
Hint: If π is a permutation, then 2 must appear in the first or second cycle. Now devise an involution that *increases* by one the parity on the number of cycles of π if 2 appears in its first cycle and *decreases* the parity otherwise.

3. Use a combinatorial argument to show that
$$\begin{bmatrix} n \\ 2 \end{bmatrix} = \frac{n!}{2} \sum_{m=1}^{n-1} \frac{1}{m(n-m)}$$

4. Prove that if $n \geq 0$ then
$$\sum_k (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k = x^{\underline{n}} \tag{1}$$

10/13 Show that the number of even permutations in S_n is same as the number of odd permutations (and hence, is equal to $n!/2$). *Hint:* Find a bijection on S_n that changes the parity of each permutation.

**Exercises from the *A Walk Through Combinatorics, 4th ed.*, Miklós Bóna, World Scientific

11/06 Find closed forms of the ordinary generating functions for each of the sequences below.

(a) $\{(-1)^n\}_{n\geq 0} = \{1, -1, 1, -1, \ldots\}$

(b) $a_{n+1} = 3a_n + 1$, where $a_0 = 1$

11/06

1. Find the partial fraction decomposition of each of the rational functions below. In other words, find A and B in each case.

   (a) $\dfrac{7x}{(2-3x)(1+2x)} = \dfrac{A}{2-3x} + \dfrac{B}{1+2x}$

   (b) $\dfrac{2x-6}{x^2-6x+4} = \dfrac{A}{x-3-\sqrt{5}} + \dfrac{B}{x-3+\sqrt{5}}$.

2. Suppose that $A(x) \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0}$ with sequence of coefficients as defined by the recursion below.

$$a_{n+2} = 5a_{n+1} - 6a_n, \quad a_0 = 2,\ a_1 = 5$$

   (a) Generate the next five terms of this sequence.

   (b) Find the closed form of $A(x)$.
   Note: As a check, you can replace $A(x)$ with your expression here and compare the coefficients of the result with your answer in part (a).

11/08

1. Find the closed form of the ordinary generating functions for each of the sequences below.

   (a) $p_n = n$

   (b) $a_{n+1} = 2a_n + n,\ a_0 = 1$ *Hint:* Part (a) may help.

   (c) $b_{n+2} = b_{n+1} + 2b_n, \quad b_0 = b_1 = 1$

2. Let B_n be a 1 by n board (one horizontal row) that consists of n squares. For $n > 0$, let b_n count the number of ways to cover B_n with a combination of 1 by 1 tiles (called *monominos*) and 1 by 2 tiles (called *dominos*) and let $b_0 = 1$. For example, the only way to cover B_1 is with one monomino, so $b_1 = 1$, but there are two ways to cover B_2: We can use either 2 monominos or 1 domino, so $b_2 = 2$. *Note:* We can illustrate this as $1 + 1$ and 2, respectively.

   (a) How many ways can you cover B_3 and B_4 using monominos and dominos?

   (b) Find a recursion equation satisfied by b_n.

3. Let $c_n = [x^n]\frac{7x}{(2-3x)(1+2x)}$. Find a closed formula for c_n. *Hint:* See Problem 1(a) from 11/06.

11/08

1. Find a closed formula for the each of the sequences defined below.

   (a) $b_{n+1} = 3b_n - 2^{n+1},\ b_0 = 3$

   (b) $g_n = [x^n]\frac{1}{1-4x+3x^2}\ \ n \geq 0$

   (c) $a_{n+2} = 5a_{n+1} - 6a_n,\ a_0 = 1, a_1 = 5$

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2.

(a) In how many ways can an n-day school year be created so that there is one holiday during the first part of the year, two holidays during the second part, and two field trips during the third part?

(b) Same as part (a) except that the field trips are *distinguishable*. For example, students could visit an art museum during the first field trip and attend a sporting event during the second field trip, or students could first attend the sporting event followed by a visit to the art museum. These should be treated as two different semesters.

11/10

1. Let $f_n = |B_n|$ as described in class (or problem 2 on 11/08 above). Give direct combinatorial proofs of each of the following identities.

   (a) For $n \geq 1$, show that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n} - 1$.

   (b) For $n \geq 0$, show that $f_0 + f_2 + \cdots + f_{2n} = f_{2n+1}$.

   (c) For $n \geq 0$, show that $\sum_{k=1}^{n} \binom{n}{k} f_{k-1} = f_{2n-1}$. *Hint:* Condition on the number of monominos that appear among the first n tiles. And remember, a tile is either a monomino or a domino.

2. In class we gave a combinatorial proof of the identity below. Answer the questions that follow.

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1, \ n \geq 0 \tag{2}$$

   (a) Use generating functions to prove (2).

   (b) Let $g_n = \sum_{k=0}^{n} f_k$. What "unexpected" recursion equation does g_n satisfy?

11/10

1. Let $f_n = |B_n|$ as described in class (or problem 2 on 11/08 above). Answer the questions below.

   (a) Find the closed form of the generating function $G(x) = \sum_{n \geq 0} f_{2n+1} x^n$.

   (b) For $n \geq 0$, show that $\sum_{k=0}^{n} (-1)^{n-k} f_{2k+1} = f_n f_{n+1}$.

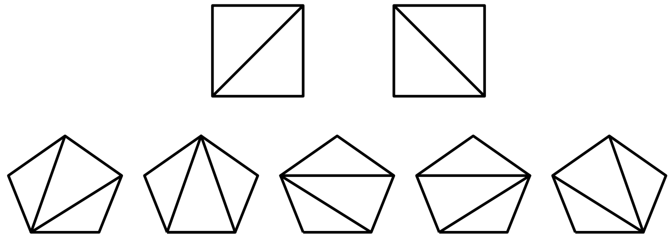

Figure 1: The 2 triangulations of a square (2+2-gon = 4-gon) and the 5 triangulations of a pentagon (5-gon).

2. Let $\{c_n\}$ be the Catalan numbers.
   (a) Show that c_n counts the number of (noncrossing) triangulations of a convex $(n+2)$-gon. See Figure 1.
   (b) In class we showed that $C(x) = \sum_{n\geq 0} c_n x^n = \frac{1-\sqrt{1-4x}}{2x}$. Use this (and the generalized Binomial Theorem) to show that $c_n = \frac{1}{1+n}\binom{2n}{n}$.
3. Let q_n count the number of ways that n votes can be counted for two candidates so that candidate A never trails candidate B. Note that we do not require an equal number of votes to be cast for each candidate.

   We can use n-strings from the alphabet $\{A, B\}$ to enumerate legal possibilities. For example, $\{A\}, \{AA, AB\}, \{AAA, AAB, ABA\}$ are the only legal strings for one, two, and three voters, respectively. If we define $q_0 = 1$, then the first 4 terms of this sequence are $1, 1, 2, 3$.
   (a) List the 6 legal strings of length 4.
   (b) Let $Q(x) = \sum_{n\geq 0} q_n x^n$. Find the closed form for $Q(x)$.
   (c) Find the closed formula for q_n.

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