Let $\pi : [n] \to [n]$ be a bijection and for $1 \le j \le n$, let $\pi_j = \pi(j)$. Then $\pi \in S_n$, that is, π is a permutation on n elements and π is usually represented in one of the following ways.

First, we have the usual *two-line* representation.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ \pi_1 & \pi_2 & \pi_3 & \cdots & \pi_{n-1} & \pi_n \end{pmatrix}$$

Using only the second line in the above representation we immediately obtain the *one-line* representation. That is,

$$\pi = (\pi_1 \ \pi_2 \ \pi_3 \ \cdots \ \pi_{n-1} \ \pi_n)$$

Another common representation of a permutation is called *cycle* notation, which is more easily explained with an example.

Example 1. Rewrite the permutations below using cycle notation.

a. $\pi = (2\ 1\ 4\ 6\ 3\ 7\ 5) \in S_7$

Then π consists of two cycles $c_1 = (12)$ and $c_2 = (34675)$ and is written

$$\pi = (12)(34675) = (34675)(12)$$

The cycle c_1 is called a *transposition* since it has length two.

b. $\delta = (4\ 2\ 1\ 3) \in S_4$ so that

$$\delta = (143)(2) = (143)$$

Notice that 2 is a fixed point since $\pi(2) = 2$ and it is customary to omit fixed points when representing permutations using cycle notation. This can be confusing at times, so we will only omit fixed points when there is a good reason to do so.

c. Finally, let $\gamma = (145)(26)(3) \in S_6$. Then using two-line notation we would write

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix}$$

To get the one-line version, we simply ignore the first row.

$$\gamma = (4\ 6\ 3\ 5\ 1\ 2)$$

Remark. The various notations have advantages. For our current task, the one-line notation will make it easier to recognize *runs* and *inversions*. Cycle notation will be the focus of section 6.1 when we introduce *Stirling Numbers of the First Kind*, i.e., cycle numbers.

One can also represent permutations using the so-called inversion table which we define below.

Definition 2. Let $\pi = (\pi_1\ \pi_2\ \pi_3\ \cdots\ \pi_{n-1}\ \pi_n) \in S_n$. Suppose that for some $i < j$ we have $\pi_i > \pi_j$. Then the pair (π_i, π_j) is called an **inversion (pair)** of π.

Now let $E(\pi)$ denote the set of all inversion pairs for the permutation π. We can construct an **inversion table** $\pi_I = b_1 b_2 \cdots b_n$ of the permutation π by letting b_j to count the number of inversion pairs (in $E(\pi)$) such that j is the second component. It is easy to see that $0 \le b_j \le n - j$. In particular, the last entry in the table must be 0 since $(j, n) \notin E(\pi)$ for any j.

Finally, define the **inversion number** $i(\pi)$ of π, by the rule

$$i(\pi) = |E(\pi)| = \sum_{k=1}^{n} b_k \tag{1}$$

For example, the pairs $(6, 3)$ and $(5, 2)$ are two 2 of the 11 inversions of the permutation $\pi = (4\ 6\ 3\ 5\ 1\ 2)$. Can you identify the other 9? Now the inversion table $\pi_I = 442010$ since

$$E(\pi) = \{(4, 3), (4, 1), (4, 2), (6, 3), (6, 5), (6, 1), (6, 2), (3, 1), (3, 2), (5, 1), (5, 2)\}$$

It turns out the we can recover the original permutation from its inversion table.

Example 3. Let $\delta_I = 2112010$. Let's try to recover the permutation δ. We begin with 7.

$$7$$

Now since 6 is out of order once, we must have

$$7\ 6$$

Notice that 5 never appears as the second entry in $E(\delta)$, so it must preceed 6 and 7.

$$5\ 7\ 6$$

Continuing

$$\begin{array}{c} 5\ 7\ 4\ 6 \\ 5\ 3\ 7\ 4\ 6 \\ 5\ 2\ 3\ 7\ 4\ 6 \\ 5\ 2\ 1\ 3\ 7\ 4\ 6 \end{array}$$

We leave it as an exercise to confirm that the inversion table for $\delta = (5\ 2\ 1\ 3\ 7\ 4\ 6)$ is $\delta_I = 2112010$.

It turns out that there is a interesting relationship between inversions and inverses. Given a permutation, we can quickly construct its inverse by using the two-line notation. For example, let $\pi = (5\ 3\ 1\ 2\ 6\ 4) \in S_6$ and switch back to the two-line format.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 2 & 6 & 4 \end{pmatrix}$$

Now swap the rows

$$\begin{pmatrix} 5 & 3 & 1 & 2 & 6 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

Finally, reorder the columns using the top row to obtain

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 6 & 1 & 5 \end{pmatrix}$$

It is easy to see that the $\pi^{-1} = (3\ 4\ 2\ 6\ 1\ 5)$. Now observe that $\pi_I = 420010$ and $\pi_I^{-1} = 221200$ so that both π and π^{-1} have the same number of inversions. In other words, $i(\pi) = i(\pi^{-1})$.

This fact was first proved by H. A. Rothe in 1800. Let $\pi = (\pi_1\ \pi_2\ \cdots\ \pi_n) \in S_n$ and construct an $n \times n$ grid P as follows. Place a dot in column k of row j whenever $\pi_j = k$. Then place $\times$ signs in all squares that have dots below and to the right of the given square. We claim that b_j is equal to the number of $\times$ signs in column j and the total number of inversions is given by the total number of $\times$ signs. Notice that the transpose of P represents π^{-1} and the placement of the $\times$ signs conforms to the procedure described above.

We illustrate with an example. Once again let $\pi = (5\ 3\ 1\ 2\ 6\ 4)$. Since $\pi_1 = 5$ we place a dot in the fifth column of the first row. Continuing, we place a dot in the third column of the second row, and so on. Now place $\times$ signs as prescribed. Figure 1 shows the resulting grid and its transpose.

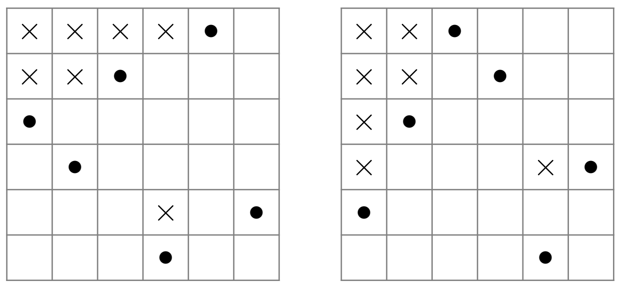

Figure 1: Inversion grid $P(\pi)$ and its transpose for the permutation π

Now let $I_n(k)$ denote the number of permutations on n elements with exactly k inversions. We note that $I_n(0) = 1$ since the identity is the only permutation with 0 inversions. It is not too hard to see that $I_n(1) = n - 1$. We also have a symmetric property

$$I_n\left(\binom{n}{2} - k\right) = I_n(k) \tag{2}$$

Proposition 4. Let $I_0(0) = 1$ and for $n \in \mathbb{N}$ and $k < 0$, let $I_n(k) = 0$. Then for $n > 0$ we have

$$I_n(k) = \sum_{j=0}^{n-1} I_{n-1}(k-j) \tag{3}$$

Proof: Let $\pi = (\pi_1\ \pi_2\ \cdots\ \pi_{n-1}) \in S_{n-1}$ and let $\pi^{(j)}$ denote the n-permutation

$$(\pi_1\ \pi_2\ \cdots\ \underbrace{n}_{j\text{th slot}}\ \cdots\ \pi_{n-1})$$

It is easy to see that $i(\pi^{(j)}) = i(\pi) + n - j$. Now suppose that $\pi \in S_{n-1}$ has m inversions with $\mathbf{Max}(0, k+1-n) \leq m \leq k$. Then $i\left(\pi^{(m+n-k)}\right) = m + n - (m + n - k) = k$. It follows that

$$\begin{aligned} I_n(k) &= \sum_{m=\mathbf{Max}(0,k+1-n)}^{k} I_{n-1}(m) \\ &= \sum_{m=k+1-n}^{k} I_{n-1}(m) \end{aligned}$$

n \ k	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0
3	1	2	2	1	0	0	0	0	0	0	0	0
4	1	3	5	6	5	3	1	0	0	0	0	0
5	1	4	9	15	20	22	20	15	9	4	1	0
6	1	5	14	29	49	71	90	101	101	90	71	49
7	1	6	20	49	98	169	259	359	455	531	573	573
8	1	7	27	76	174	343	602	961	1415	1940	2493	3017

Table 1: Permutations on $[n]$ with k inversions

since $I_{n-1}(m) = 0$ whenever $m < 0$. Now after the substitution $m = k - j$, the last expression is equal to the right-hand side of (3). □

With the help of (4), we can now generate a Pascal type triangle for $I_n(k)$. Table 1 displays the first 8 rows of such a table. Notice that we have the recursion formula below, which is valid for $k < n$ (below the zig-zag border).

$$I_n(k) = I_n(k-1) + I_{n-1}(k), \quad k < n \tag{4}$$

We list just two of the numerous formulas involving inversion numbers.

$$I_n(2) = \binom{n}{2} - \binom{n}{0}, \quad n \geq 2$$

$$I_n(3) = \binom{n+1}{3} - \binom{n}{1}, \quad n \geq 3$$

We wish to define another important property associated with permutations and inversion numbers, parity.

Definition 5. Let $\pi \in S_n$ be a permutation. Define the map sgn: $S_n \longrightarrow \{-1, 1\}$ by the rule $\operatorname{sgn}(\pi) = (-1)^{i(\pi)}$. Then π is called an *even* permutation if $\operatorname{sgn}(\pi) = 1$ and an *odd* permutation if $\operatorname{sgn}(\pi) = -1$. In other words, even permutations have an even number of inversions and odd permutations have an odd number of inversions. We shall call sgn the *parity* operator.

Theorem 6. Let $\alpha, \beta \in S_n$ be permutations and let sgn be parity operator defined above. Then

$$\operatorname{sgn}(\alpha\,\beta) = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)$$

In other words, the sgn operator is multiplicative.

Example 7. Let $\alpha, \beta \in S_7$, with $\alpha = (24156)(37)$ and $\beta = (1)(2)(35)(4)(6)(7) = (35)$. Notice that we are using cycle notation and, at least in this example, we are suppressing the fixed points of β. We leave it as an exercise to confirm that $i(\alpha) = 13$ and $i(\beta) = 3$. Hence α and β are odd permutations. On the other hand,

$$\alpha\beta = (5\ 4\ 6\ 1\ 7\ 2\ 3)$$

so that

$$E(\alpha\beta) = \{54, 51, 52, 53, 41, 42, 43, 61, 62, 63, 72, 73\}$$

and

$$i(\alpha\beta) = 12$$

In other words, $\alpha\beta$ is even, as expected.

Generating Functions

Now for a fixed n, let $J_n(x)$ denote the ordinary power series generating function for $\{I_n(k)\}_{k\geq 0}$. That is, let $J_n(x) = \sum_{k\geq 0} I_n(k)x^k$. Then by Proposition 4,

$$\begin{aligned} J_n(x) &= \sum_{k\geq 0} I_n(k)x^k \\ &= \sum_{k\geq 0}\sum_{j=0}^{n-1} I_{n-1}(k-j)x^k \\ &= \sum_{j=0}^{n-1}\sum_{k\geq 0} I_{n-1}(k-j)x^{k-j}\,x^j \\ &= J_{n-1}(x)\sum_{j=0}^{n-1} x^j \end{aligned}$$

In other words, $J_n(x)$ satisfies the recursion equation

$$J_n(x) = (1 + x + x^2 + \cdots + x^{n-1})J_{n-1}(x)$$

Now since $J_1(x) = 1$, we have shown

Theorem 8.

$$\begin{aligned} J_n(x) &= (1 + x + x^2 + \cdots + x^{n-1})\cdots(1 + x)(1) & (5) \\ &= \frac{(1 - x^n)(1 - x^{n-1})\cdots(1 - x^2)(1 - x)}{(1 - x)^n} & (6) \end{aligned}$$

Exercises

1. Let $\pi = (146)(23)(57) \in S_7$.

   (a) Rewrite π using one-line notation and find the inversion table for π. *Hint:* It might be easier to convert from cycle notation to two-line notation first.

   (b) Find the inversion table for π^{-1}. Also, confirm that $i(\pi) = i(\pi^{-1})$.

2. Let $\sigma = (5\ 4\ 2\ 3\ 1)$. Find the set of inversions of σ, $E(\sigma)$. Also, compute the inversion grid $P(\sigma)$ (see Figure 1). *Explain why the grid construction works.*

3. Let $\delta_I = 4102110 \in I_7$ be an inversion table for some $\delta \in S_7$. Find δ.

4. Let $\pi = (\pi_1\ \pi_2\ \cdots\ \pi_n) \in S_n$ be a permutation and let $E(\pi)$ be the set of its inversions.

   (a) Prove that $E(\pi)$ is transitive. That is, prove that if (a, b) and (b, c) are in $E(\pi)$, then $(a, c) \in E(\pi)$.

   (b) Conversely, let E be any transitive subset of $T = \{(x, y) : 1 \leq y < x \leq n;\ x, y \in [n]\}$ whose complement $E^c = T \setminus E$ is also transitive. Prove that there is a permutation π such that $E(\pi) = E$.

5. Let $\pi \in S_n$ and let $\pi_I = b_1 b_2 \ldots b_n$ be its inversion table. Now suppose that $\sigma_I = b'_1 b'_2 \ldots b'_n$ where $b'_j = n - j - b_j$. Find σ.

6. Prove the symmetric property
$$I_n\left(\binom{n}{2} - k\right) = I_n(k) \tag{7}$$

7. **Musical Chairs** In the traditional game, n children walk around a circle of $n - 1$ chairs while music is playing. When the music stops, players race to sit in the nearest chair. The child left standing is eliminated, a chair is removed and the game continues until one player is left standing. In our combinatorial variation, we arrange n students sitting in a circle and then ask every mth student to leave the circle until no student remains. The order in which the students leave the circle defines the permutation. For example, when $n = 8$ and $m = 3$, we obtain the permutation $\pi = (3\ 5\ 1\ 7\ 4\ 2\ 8\ 6)$ and the inversion table $\pi_I = 24020200$. In this case, the third student is the first to leave the circle, the sixth student is the second to leave, and so on. It turns out the the seventh student is the last to leave the circle. *Note:* Our game is a less violent version of the classical Josephus problem.

   Find a simple recurrence relation for the elements $b_1 b_2 \ldots b_n$ of the inversion table of the resulting permutation for arbitrary n and m.

8. The inversion table that we defined in Definition 2 could also be called a left-inversion table. It turns out that several others can be defined. So let $\pi = (\pi_1\ \pi_2\ \cdots\ \pi_n) \in S_n$ and let $E(\pi)$ be the set of inversion pairs as before. We define the **right-inversion** table $\pi_R = c_1c_2 \dots c_n$, where c_j counts the number of inversions whose first component is j. Notice that $0 \leq c_j < j$. For example, if $\pi = (4\ 6\ 3\ 5\ 1\ 2)$ then $\pi_I = 442010$, as we saw before, and $\pi_R = 002324$.

   Now let $C_j = c_{\pi_j}$. Prove that $\pi \in S_n$ is an involution if and only if $b_j = C_j$ for $1 \leq j \leq n$.