1. (6 points) What is the largest coefficient in the expansion of $(x+y+z+w)^6$? EXPRESS YOUR ANSWER AS AN INTEGER. *A bit of trial and error is probably required.*

**Solution:**

The coefficients have the form $\binom{6}{a,b,c,d}$ where $a+b+c+d=6$. After a bit of trial and error, it's easy to see that largest coefficient must be

$$\binom{6}{2,2,1,1} = \frac{6!}{2!2!} = 180$$

2. (4 points) Find the sum below. JUSTIFY YOUR RESPONSE.

$$\sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k}$$

**Solution:**

By the Multinomial Theorem,

$$\begin{aligned} k^n &= \left(\sum_{j=1}^{k} 1\right)^n \\ &= \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} 1^{n_1} 1^{n_2} \cdots 1^{n_k} \\ &= \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} \end{aligned}$$

3. (10 points) Give a combinatorial proof of the identity below.

$$4^k = \sum_{j=0}^{k} 3^j \binom{k}{j}$$

**Solution:**

**Question -** How many $k$-words are there using the alphabet $\mathcal{A} = \{a, b, c, d\}$?

For the left-hand side, there are 4 choices for the first letter, 4 choices for the second, and so on. So by the product rule, there are $4 \times 4 \times \cdots \times 4 = 4^k$ such words.

For the right-hand side, we condition on the location of the letter $a$. So there are $\binom{k}{k-j} = \binom{k}{j}$ ways to position $k - j$ occurrences of the letter $a$ and the remaining positions may be filled with letters from the alphabet $\{b, c, d\}$ in $3^j$ ways. So by the product rule, there are $\binom{k}{j}3^j$ ways to create a $k$-word with exactly $k - j$ occurrences of $a$. Since these collections are disjoint and since a $k$-word contains zero $a$'s, or one $a$, or two $a$'s, etc., we may sum as $j$ ranges from 0 to $k$. The result follows.