1. (6 points) Let $\mathcal{P}_n$ denote the collection of all (set) partitions on $[n]$.

   (a) Let $\sigma = 17/245/38/6 \in \mathcal{P}_8$ . Rewrite $\sigma$ in canonical form.

   **Solution:**

   Following our notation in class, the canonical form is $w(\sigma) = 12322413$.

   (b) Let $\delta \in \mathcal{P}_{10}$ and let $\omega(\delta) = 1213222145$ be its canonical form. Find $\delta$ (in standard block form).

   **Solution:**

   138/2567/4/9/10

2. (a) (3 points) How many set partitions of [9] exclude blocks with the singleton element 9? For example, the partition 14/25/3678/9 should **not** be counted. YOU MAY FIND THE TABLE ON PAGE 2 USEFUL.

   **Solution:**

   Should be $17007 = b_9 - b_8$, where $b_n$ are the Bell numbers.

   (b) (4 points) State and prove a general rule. That is, how many set partitions of $[n]$ exclude blocks with the singleton element n?

   **Solution:**

   Claim that the number of (set) partitions of $[n]$ that exclude singleton blocks of $n$ is given by $b_n - b_{n-1}$.

   To see this, observe that if a partition of $[n]$ includes a singleton block with $n$, then there are $b_{n-1}$ ways to partition the remaining elements (as we saw in the proof of Theorem 5.12 in the text). It follows that the remaining partitions are the ones that we are looking for. Since the total number of partitions of $[n]$ is $b_n$, it follows that $b_n - b_{n-1}$ yields the desired count.

   *Note:* This is easy to confirm. For example, $\mathcal{P}_3 = \{1/2/3, 12/3, 1/23, 13/2, 123\}$ and the last $3 = b_3 - b_2$ are the only partitions that exclude the singleton 3.

   *Remark:* It should be pretty clear that there is nothing special about the element 9 in part (a). For example, there are 17007 partitions that exclude 8. Now the interesting question becomes, how many partitions of [9] contain no singleton blocks?

3. (7 points) Find the number of compositions of $n$ into an even number of parts.

**Solution:**

If $n = 1$ then there are zero compositions with an even number of parts. If $n > 1$, then we claim that half of all compositions have an even number of parts. In other words, $2^{n-1}/2 = 2^{n-2}$. To see this recall that the number of compositions of $n$ into $k$ parts was given by $\binom{n-1}{k-1}$ and the total number of compositions is

$$2^{n-1} = \sum_k \binom{n-1}{k-1} \tag{1}$$

Now

$$\begin{aligned} 0 &= \sum_k \binom{n-1}{k-1}(-1)^k \qquad (2) \\ &= \sum_{\substack{k \\ k \text{ even}}} \binom{n-1}{k-1}(-1)^k + \sum_{\substack{k \\ k \text{ odd}}} \binom{n-1}{k-1}(-1)^k \\ &= \sum_{\substack{k \\ k \text{ even}}} \binom{n-1}{k-1} - \sum_{\substack{k \\ k \text{ odd}}} \binom{n-1}{k-1} \end{aligned}$$

Thus

$$\sum_{\substack{k \\ k \text{ even}}} \binom{n-1}{k-1} = \sum_{\substack{k \\ k \text{ odd}}} \binom{n-1}{k-1}$$

In other words, half of the compositions have an even number of parts. So by (1), we get $2^{n-2}$.