1. (6 points) Let $\sigma \in S_7$ with inversion table $\sigma_I = 3002110$.

   (a) Rewrite σ using one-line notation.

   Solution:

   $\sigma = (2\ 3\ 7\ 1\ 5\ 4\ 6)$

   (b) Rewrite σ using cycle notation.

   Solution:

   $\sigma = (123764)(5)$

2. (7 points) Let $\pi = (\pi_1\ \pi_2\ \cdots\ \pi_n) \in S_n$ be a permutation and let $E(\pi)$ be the set of its inversions. Prove that $E(\pi)$ is transitive. That is, prove that if (a, b) and (b, c) are in $E(\pi)$, then $(a, c) \in E(\pi)$.

   Solution:

   This is rather straight-forward. If (a, b) and (b, c) are in $E(\pi)$, then $a > b$ and $b > c$, hence $a > c$. Now if π is written in the usual one-line notation, a lies to the left of b and b lies to the left of c. In other words, a lies to the left of c and so $(a, c) \in E(\pi)$.

3. (7 points) A permutation is called even (resp. odd) if its inversion number is even (resp. odd). For example, the permutation σ in Problem 1 is odd since $i(\sigma) = |E(\sigma)| = 7$. Prove that if $\pi \in S_k$ has only *one* cycle, then π is even if and only if k is odd.

Solution:

In other words, if $\sigma \in \left[{[n] \atop 1}\right]$, then σ is even if n is odd, and σ is odd whenever n is even. Observe that this is obviously true when $n = 1$, since $\left[{[1] \atop 1}\right] = \{(1)\}$ and the identity permutation has zero inversions, in other words, it's an even permutation. Now when $n = 2$, we have $\left[{[2] \atop 1}\right] = \{(12)\}$. But $(12) = (2\ 1)$ clearly has one inversion, so that (12) is odd.

We use cycle notation for the rest of this proof. Now every permutation in $\left[{[n] \atop 1}\right]$ is obtained by inserting n into any one of the $n-1$ positions of $\pi = (\pi_1\pi_2\pi_3 \cdots \pi_{n-1})$ for some $\pi \in \left[{[n-1] \atop 1}\right]$. Now by Theorem 6 on the Inversion handout, $\operatorname{sgn}(\pi \cdot (n)) = \operatorname{sgn}(\pi)\operatorname{sgn}((n)) = \operatorname{sgn}(\pi) \cdot 1 = \operatorname{sgn}(\pi)$. In other words, the one-cycle π and the two-cycle $\pi \cdot (n)$ have the same parity.

Now let $\pi^{(j)} = (\pi_1\pi_2\pi_3 \cdots \pi_j\, n\, \pi_{j+1} \cdots \pi_{n-1}) \in \left[{[n] \atop 1}\right]$. In other words, $\pi^{(j)}$ is the one-cycle in S_n obtained by inserting n into the one-cycle π (described above) in the indicated position. We claim that $\pi^{(j)} = \pi \cdot (n) \cdot (\pi_j n)$. Now if the claim is true, then

$$\begin{aligned}\operatorname{sgn}(\pi^{(j)}) &= \operatorname{sgn}(\pi \cdot (n) \cdot (\pi_j n)) = \operatorname{sgn}(\pi)\operatorname{sgn}((\pi_j n)) \\ &= \operatorname{sgn}(\pi)(-1)\end{aligned}$$

In other words, one-cycles in S_{n-1} and S_n have different parities. And since the parity of a one-cycle in S_1 is even, the result follows by induction.

To prove the claim, observe that $\pi^{(j)}[\pi_j] = n$ and $\pi^{(j)}[n] = \pi_{j+1}$. But

$$\pi \cdot (n) \cdot (\pi_j n)[\pi_j] = n$$

and

$$\begin{aligned}\pi \cdot (n) \cdot (\pi_j n)[n] &= \pi \cdot (n)[\pi_j] \\ &= (\pi_1\pi_2\pi_3 \cdots \pi_{n-1})[\pi_j] \\ &= \pi_{j+1}\end{aligned}$$

and the claim is proven.

Remark: Let $\pi = (1)(2)\cdots(j-1)(j+1)\cdots(k-1)(k+1)\cdots(n)(jk)$ for some $1 \le j < k \le n$. So π has $n-2$ fixed points and $\pi[j] = k$ and $\pi[k] = j$. Such permutations are called *transpositions* because they swap (transpose) the entries j and k and leave everything else alone. In such cases, we often omit the fixed points and simply write $\pi = (jk)$. It turns out that the parity of a transposition is always odd, a fact that we exploited above when we stated $\operatorname{sgn}((\pi_j n)) = -1$. To see this, we use two-line notation and construct $E((jk))$.

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & j & \cdots & k & \cdots & n-1 & n \\ 1 & 2 & \cdots & k & \cdots & j & \cdots & n-1 & n \end{pmatrix}$$

Then there are $k - j$ inversion pairs whose left entry is k and $k - (j+1)$ inversion pairs whose right entry is j. It follows that $|E((jk))| = 2k - 2j - 1$ which is odd, as expected.

We illustrate all of this with an example. Let $\pi = (1365742) \in \left[{[7] \atop 1}\right]$. Then

$$\begin{aligned} \pi &= (1365742) \\ &= (136542)(57) \end{aligned}$$

so that

$$\begin{aligned} \operatorname{sgn}(\pi) &= \operatorname{sgn}((136542))\operatorname{sgn}((57)) \\ &= \operatorname{sgn}((136542))(-1) \end{aligned}$$

In other words, π and $(136542) \in \left[{[6] \atop 1}\right]$ have different parities. Notice that $(136542)(57)[5] = 7 = \pi[5]$ and $(136542)(57)[7] = (136542)[5] = 4 = \pi[7]$ as expected.