1. (10 points) A local restaurant sent out 38 tablecloths to be laundered. Each tablecloth had zero or more stains from coffee (C), vinegar (V), grease (G), or pizza sauce (S) as indicated in the table below.

Stain	Count	Stain	Count
V	21	G	19
S	20	C	12
VC	6	VG	12
SG	10	CS	5
CG	7	VS	11
VCS	3	VCG	4
VGS	6	CSG	3

If two of the tablecloths had all 4 stains, answer the question below.

(a) How many tablecloths had no stains?

Solution:

Lets first construct $N(x)$. According to the table,

$$N(x) = 38 + 72x + 51x^2 + 16x^3 + 2x^4$$

It follows that

$$\begin{aligned} E(x) &= N(x-1) \\ &= 3 + 10x + 15x^2 + 8x^3 + 2x^4 \end{aligned}$$

Now it's easy to see that there are exactly 3 shirts with no stains.

(b) How many tablecloths had exactly one stain?

Solution:

From part (a), we see that the number of shirts with exactly one stain is $e_1 = [x^1]E(x) = 10$.

2. (10 points) In how many ways can the 12 letters $\{a^5, b^4, c^3\}$ be arranged so that there are not 5 consecutive a's, nor 4 consecutive b's, nor 3 consecutive c's. For example, *abcccababababa* and *caaaaacbbbbc* are forbidden strings, but the string *aaaabbbccabc* is legal. *Note:* There are several ways to compute this. I used the principle of inclusion and exclusion.

Solution:

Following the hint, let $p_x,\ x \in \{a, b, c\}$ correspond to the property that all identical letters are consecutive. Now let N count the number of distinguishable permutations of the 12 letters, then $N = 12!/5!/4!/3! = 27720$.

What is $N(p_a)$? To answer this, we treat the a's as a single block, call it A. So how many ways are there to arrange the 8 letters $\{A, b^4, c^3\}$. Once again, we appeal to our formula for distinguishable permutations to conclude that

$$N(p_a) = \frac{8!}{1!4!3!} = 280$$

Similarly

$$N(p_b) = \frac{9!}{5!1!3!} = 504 \text{ and } N(p_c) = \frac{10!}{5!4!1!} = 1260$$

Using similar arguments, it follows that

$$N(p_a p_b) = \frac{5!}{3!} = 20$$
$$N(p_a p_c) = \frac{6!}{4!} = 30$$
$$N(p_b p_c) = \frac{7!}{5!} = 42$$
$$N(p_a p_b p_c) = 3! = 6$$

It follows by PIE that the number of permissable strings is

$$e_0 = 27720 - (280 - 504 - 1260) + (20 + 30 + 42) - 6 = 25762$$