1. (12 points) Consider the recursion below and answer the questions that follow.

$$a_{n+2} = 3a_{n+1} - a_n, \quad a_0 = 1, \; a_1 = 4 \tag{1}$$

(a) *Carefully* find the next 3 terms in this sequence.

Solution:

$a_2 = 3(4) - 1 = 11$, etc., so the next 3 terms are 11, 29, 76. Here are the first 11 terms.

(b) Find the closed form of the generating function for this sequence. That is, find the closed form of $A(x) = \sum_{n\geq 0} a_n x^n$.

Solution:

We can multiply the recurrence in (1) by x^{n+2} and sum over n to produce

$$\sum_{n\geq 0} a_{n+2}x^{n+2} = 3\sum_{n\geq 0} a_{n+1}x^{n+2} - \sum_{n\geq 0} a_n x^{n+2}$$

This is equivalent to

$$A(x) - a_0 - a_1 x = 3x(A(x) - a_0) - x^2 A(x)$$

Note: One can also use the Wilf rules to produce the same equation.

Now apply the initial conditions and rearrange to obtain

$$A(x)(1 - 3x + x^2) = 1 + 4x - 3x$$

Thus

$$A(x) = \frac{1 + x}{1 - 3x + x^2}$$

Compare the coefficients in the Taylor Series expansion of $A(x)$ with the terms listed in part (a).

2. (8 points) Let $\{b_n\}_{n\geq 0}$ be the sequence defined by the recursion below.

$$b_{n+3} = 3b_{n+2} - b_{n+1} + 4b_n, \quad b_0 = 1,\ b_1 = 5,\ b_2 = 12 \tag{2}$$

According to a theorem mentioned in class, the closed form of the ordinary generating function whose sequence of coefficients satisfies (2) must be of the form

$$B(x) = \sum_{n\geq 0} b_n x^n = \frac{q(x)}{1 - 3x + x^2 - 4x^3} \tag{3}$$

where $q(x)$ is a nonzero polynomial of degree strictly less than 3.

Find the closed form of $B(x)$. In other words, find $q(x)$.

Solution:

Although we could repeat the method we used in Problem 1 or use the Wilf rules, we'll follow the suggestion above. Notice that if $B(x)$ is as shown in (3), then $q(x) = a + bx + cx^2$. So we just need to find $a, b,$ and c. It follows by Taylor's Theorem that

$$1 = b_0 = \frac{B(0)}{0!} = a$$
$$5 = b_1 = \frac{B'(0)}{1!} = 3 + b \implies b = 2$$
$$12 = b_2 = \frac{B''(0)}{2!} = 14 + c \implies c = -2$$

It follows that

$$q(x) = 1 + 2x - 2x^2$$

and

$$B(x) = \frac{1 + 2x - 2x^2}{1 - 3x + x^2 - 4x^3}$$

Once again, we can use an external tool to compute the first few terms of the Taylor series expansion of $B(x)$ as a sanity check. Notice that

$$b_3 = 3(12) - 5 + 4(1) = 35 = [x^3]B(x)$$
$$b_4 = 3(35) - 12 + 4(5) = 113 = [x^4]B(x)$$

as expected.