1. (12 points) Let $G(x) = \sum_{n\geq 0} g_n x^n$ and suppose that

$$G(x) = \frac{1+x}{(1+2x)(1-3x)} = \frac{1+x}{1-x-6x^2} \tag{1}$$

(a) The sequence of coefficients $\{g_n\}$ satisfies a linear recursion equation of the form $g_{n+2} = A\, g_{n+1} + B\, g_n$. Find this equation and state the initial conditions.

Solution:

As we saw in class, the recursion should be

$$g_{n+2} = g_{n+1} + 6g_n$$

See the denominator of the right-hand side of (1). For the initial conditions, we have

$$g_0 = \frac{G(0)}{0!} = 1$$
$$g_1 = \frac{G'(0)}{1!} = 2$$

(b) Find the closed formula for the sequence of coefficients, g_n.

Solution:

A routine partial fraction decomposition yields

$$G(x) = \frac{1+x}{(1+2x)(1-3x)} = \frac{1/5}{1+2x} + \frac{4/5}{1-3x}$$

It follows that

$$G(x) = \frac{1}{5}\sum_{n\geq 0}(-2)^n x^n + \frac{4}{5}\sum_{n\geq 0} 3^n x^n$$

Thus

$$g_n = [x^n]G(x) = \frac{(-2)^n + 4\cdot 3^n}{5}$$

Notice that with this formula, we can confirm that $g_0 = 1$ and $g_1 = 2$, as we stated in part (a).

2. (8 points) Consider the sequence below and answer the questions that follow.

$$\{h_n\} = \{4, 1, 5, 6, 11, 17, 28, 45, 73, 118, \dots\} \tag{2}$$

(a) The first 10 terms in the sequence defined in (2) appear to satisfy a familiar linear recurrence equation. Find it.

Solution:

$$h_{n+2} = h_{n+1} + h_n$$

(b) Find the closed form of the generating function $H(x) = \sum_n h_n x^n$.

Solution:

We already know that the ogf will be of the form

$$H(x) = \frac{4 + bx}{1 - x - x^2} \quad \text{(why?)}$$

But

$$1 = h_1 = \frac{H'(0)}{1!} = 4 + b \implies b = -3$$

Thus

$$H(x) = \frac{4 - 3x}{1 - x - x^2}$$