Date	**Section**	**Exercises**** (QC - Quick Check and CE - Class Exercises)
02/19*	-	See below.
02/21*	-	See below.
02/23*	-	See below.
03/04*	-	See below.
03/06*	Notes	1, 2 from here. Also, see below.
03/08*	Notes	3, 4, 5 from here. Also, see below.
03/11*	-	See below.
03/13*	-	See below.
03/15*	-	See below.
03/18*	-	See below.
03/20*	-	(Optional) 4 from here. Also, see below.
03/22*	16.2	QC - 3; CE - 5, 6; Also, see below.
03/25*	-	CE - 43. Also, see below.
03/27*	-	CE - 31 and read Dilworth's theorem. Also, see below.
03/29*	-	See below.
04/01*	-	CE - 5, 32-34. Also, see below.
04/03*	-	See below.
04/08*	Notes	Exercises 15 and 16 from Chapter 2. Also, see below.
04/10*	-	See below.
04/15*	-	See below.

02/19

1. Consider the following orthogonality identity.

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{n-k} = \delta_n(m) \tag{1}$$

   (a) There is a symmetric version of (1). State it.

   (b) Use the Stirling Inversion Theorem (Theorem 2 here) to prove (1).

   (c) In Math 481 we proved (2). See Example 5 here.

$$x^n = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} x^{\underline{k}} \tag{2}$$

   We also proved the next result. See (7) here.

$$x^{\overline{n}} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k \tag{3}$$

   Now use (2) to prove the following

$$x^n = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} (-1)^{n-k} x^{\overline{k}} \tag{4}$$

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(d) Use the identities (3) and (4) to prove (1).

(e) Now use (1) (or part (a)) to prove the Stirling Inversion Theorem.

2. Reprove the Binomial Inversion Theorem (Equation (2) here) as indicated below.
   (a) Let $f(x) = \sum_n f_n\, x^n/n!$ and $g(x) = \sum_n g_n\, x^n/n!$ and mimic the proof of Theorem 2 shown here.

   (b) Let $f(x) = \sum_n f_n\, x^n$ and $g(x) = \sum_n g_n\, x^n$ and once again mimic the proof of Theorem 2 shown here.

02/21

1. Show that

$$x^{\overline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{\underline{n}} \tag{5}$$

   and

$$x^{\underline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} x^{\overline{n}} \tag{6}$$

2. Prove that

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \sum_j \begin{bmatrix} n \\ j \end{bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix} \tag{7}$$

3. If $n \geq k \geq 1$, prove that

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \binom{n-1}{k-1} \frac{n!}{k!} \tag{8}$$

02/23

1. Find a combinatorial proof of (7) from 02/21.
   Hint: $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]$ counts the number of ways to seat n knights at j nonempty round tables and $\left\{\begin{smallmatrix} j \\ k \end{smallmatrix}\right\}$ counts the number of ways to distribute these j tables into k nonempty rooms. Both the tables and rooms are indistinguishable.

2. Find a combinatorial proof of

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^k = (-1)^n \delta_n(m)$$

   Hint: Using the hint given in the previous exercise, let $\mathcal{E}$ contain all seating arrangements with an even number of tables and let $\mathcal{O}$ contain all seating arrangements with an odd number of tables. Now find a bijection between $\mathcal{E}$ and $\mathcal{O}$ that has two exceptions.

3. Prove that
$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{0<j_1<j_2<\cdots<j_{n-k}<n} j_1 j_2 \cdots j_{n-k}$$
Hint: Divide both sides of (3) by x and notice that the left-hand side is the product $(x+1)(x+2)\cdots(x+n-1)$. Now compare the coefficient of x^{k-1} on the left and right-hand sides of the resulting identity.

4. Referring to Example 3 here.
   (a) Verify equations (9) and (13).

   (b) Prove that
$$\frac{k}{n}\binom{n}{k} + \frac{k+1}{n}\binom{n}{k+1} = \binom{n}{k}$$

5. Use LIF to show that
$$b_n = \sum_k \binom{k}{n-k} a_k \quad \text{iff} \quad a_n = \frac{1}{n}\sum_k \binom{2n-k-1}{n-k} k b_k (-1)^{n-k}$$
Hint: Follow Example 3 from here.

03/04

1. Let $f(x) = \sum_{n\geq 1} f_n\, x^n \in x\,\mathbb{C}[[x]],\ f_1 \neq 0$. For any $g(x) \in \mathbb{C}((x))$, define the degree of $g(x)$ as we did for formal power series. That is, $\deg(g(x)) = \min\{n \in \mathbb{Z} \,|\, [x^n]g(x) \neq 0\}$. Now let $k > 0$. Show that $f(x)^{-k} \in \mathbb{C}((x))$ with $\deg(f(x)^{-k}) = -k$.

2. Confirm the (**) step in the first proof of LIF from today's lecture.

03/06

1. Suppose that $z = z(x)$ satisfies $z = x\phi(z)$. For $n \geq 0$, show that
$$[z^n]\phi(z)^n = [x^n]\left\{\frac{xz'(x)}{z(x)}\right\} = [x^n]\frac{1}{1-x\phi'(z(x))} \tag{9}$$

   Solution:

   The direct proof is routine. As an alternative, we have
$$\begin{aligned}[z^n]\phi(z)^n &= [z^{n-1}]\frac{1}{z}\,\phi(z)^n \\ &= n[x^n]\int \frac{dy}{y}\,\Big|_{y=z(x)}\end{aligned}$$
   where we invoked the Lagrange Inversion formula backwards. And we can proceed as we did for (13) in Problem 03 below.

2. Let $g_n = [x^n](1+x+x^2)^n,\ n \geq 0$. Use the previous exercise to show that

$$g_n = [x^n]\frac{1}{\sqrt{1-2x-3x^2}} \tag{10}$$

Solution:

3. Show the following. *Hint:* For (11) use the generalized Binomial theorem.

$$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0} \binom{2n}{n} x^n \tag{11}$$

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k = \sum_{n\geq 0} \frac{k(2n+k-1)!}{n!(n+k)!} x^n \tag{12}$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k = \sum_{n\geq 0} \binom{2n+r}{n} x^n \tag{13}$$

Solution: For (11) we have

$$\frac{1}{\sqrt{1-4x}} = (1+(-4x))^{-1/2} = \sum_{n\geq 0} \binom{-1/2}{n} (-4x)^n = \cdots$$

We leave the details to the student.

For (12), we let $C(x) = (1-\sqrt{1-4x})/(2x)$ and let $z(x) = C(x) - 1$. Then as we have shown before (see Example 2),

$$z = x(1+z)^2 = x\phi(z) \tag{14}$$

Now let $W(z) = (1+z)^k$, then by the Lagrange Inversion formula

$$\begin{aligned}
[x^n]C(x)^k &= [x^n]W(z(x)) \\
&= \frac{1}{n}[z^{n-1}]W'(z)\phi(z)^n \\
&= \frac{k}{n}[z^{n-1}](1+z)^{k-1}(1+z)^{2n} \\
&= \frac{k}{n}[z^{n-1}](1+z)^{2n+k-1} \\
&= \frac{k}{n}\binom{2n+k-1}{n-1}
\end{aligned}$$

For (13), we once again use the Lagrange Inversion formula (step (*) below), but in the reverse direction. Let $z(x)$, $C(x)$, and $\phi(z)$ be as shown above and let $g(x) = \sum_{n\geq 0}\binom{2n+r}{n} x^n$. Then

$$\begin{aligned}
[x^n]g(x) = \binom{2n+r}{n} &= [z^n](1+z)^{2n+r} \\
&= [z^{n-1}]\frac{(1+z)^r}{z}(1+z)^{2n} \\
&= [z^{n-1}]\frac{(1+z)^r}{z}\phi(z)^{2n} \\
&\overset{*}{=} n[x^n]\int \frac{(1+y)^r}{y}\,dy \Big|_{y=z(x)} \\
&= [x^n]xD_x \int \frac{(1+y)^r}{y}\,dy \Big|_{y=z(x)} \\
&= [x^{n-1}]\frac{(1+z)^r}{z}\frac{dz}{dx}\Big|_{z=x\phi(z)}
\end{aligned} \tag{15}$$

Now by (14),

$$\frac{dz}{dx} = \phi(z) + x\phi'(z)\,\frac{dz}{dx}$$

Rearranging produces

$$\frac{dz}{dx} = \frac{\phi(z)}{1 - x\phi'(z)}$$

Inserting this into (15) yields

$$\begin{aligned}
\binom{2n+r}{n} &= [x^{n-1}]\frac{(1+z)^r}{z}\,\frac{\phi(z)}{1-x\phi'(z)}\,\bigg|_{z=x\phi(z)} \\
&= [x^{n-1}]\frac{\phi(z)}{z}\,\frac{(1+z)^r}{1-x\phi'(z)}\,\bigg|_{z=x\phi(z)} \\
&= [x^{n-1}]\frac{1}{x}\,\frac{(1+z)^r}{1-x\phi'(z)}\,\bigg|_{z=x\phi(z)} \\
&= [x^{n}]\frac{(1+z)^r}{1-x\phi'(z)}\,\bigg|_{z=x\phi(z)}
\end{aligned}$$

Now since $\phi'(z) = 2(1+z)$ and since $1 + z(x) = C(x)$, the last expression above produces

$$\begin{aligned}
\binom{2n+r}{n} &= [x^n]\frac{C(x)^r}{1-2xC(x)} \\
&= [x^n]\frac{C(x)^r}{\sqrt{1-4x}}
\end{aligned}$$

which is equivalent to (13).

03/08

1. Let $M_0 = 1$ and for $n > 0$, suppose that

$$M_n = M_{n-1} + \sum_{k=2}^{n} M_{k-2}M_{n-k} \tag{16}$$

Show that if $M(x) = \sum_{n\geq 0} M_n\, x^n$, then $M(x)$ satisfies the functional equation

$$M(x) - 1 = xM(x) + x^2M(x)^2 \tag{17}$$

03/11

1. Let $\{a_n\}_{n\geq 0} \subset \mathbb{R}$ with $a_0 \neq 0$. Find a sum formula for $[z^n]\left(\sum_{k=0}^{N} a_k\, x^k\right)^n$ when $N \in \{2, 3\}$. Do you see a pattern?

2. Let $\mathcal{T} = \mathcal{T}^{\Omega}$ where $\Omega = \{0, 1, 3\}$. However, this time we measure the size of each tree by the number of edges. Let $T(x)$ be the ordinary generating function for $\mathcal{T}$. Find a sum formula for $[x^n]T(x)$.

3. Let m_n be the Motzkin numbers as defined here and let $\{c_n\}_{n\geq 0}$ be the Catalan numbers. Answer the questions below.

   (a) Show that
   $$c_n = m_{2n} \tag{18}$$

   (b) Show that
   $$m_n = \sum_k \binom{n}{2k} c_k \quad \text{and} \quad c_{n+1} = \sum_k \binom{n}{k} m_k$$

   (c) Show the Motzkin's original definition (stated here) is equivalent to the one given in class by showing that the original definition satisfies the following recursion.
   $$m_n = m_{n-1} + \sum_{k=2}^{n} m_{k-2} m_{n-k}, \quad n > 0$$

4. Find a formula t_n for the number of triangulations of an $(n+2)$-gon. So $t_1 = 1$ and $t_2 = 2$ since there is one triangulation of a triangle and there are two triangulations of a square.

03/13

1. Consider the *lattice of compositions*, $(K_n, \leq)$. Here K_n is the set of all compositions of n and $\alpha \leq \beta$ is a refinement of compositions defined by
   $$\text{If } [\alpha_1, \alpha_2, \ldots, \alpha_p] \vDash \alpha \text{ and } [\beta_1, \beta_2, \ldots, \beta_q] \vDash \beta, \text{ then } [\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_l}] \vDash \beta_k \text{ for } k \in [q].$$
   For example, in K_{11}, $3+2+5+1$ is a refinement of $5+5+1$ hence $[3,2,5,1] \leq [5,5,1]$. On the other hand, $[3,3,4,1] \not\leq [5,5,1]$. Sketch the Hasse diagram for K_4.

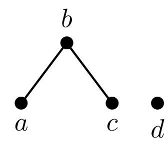
2. The *Young lattice* $(Y, \leq)$ is the set of all integer partitions and $\alpha \leq \beta$ if the Young diagram for α is a contained in the Young diagram for β. Sketch the Hasse diagram for Y up to integer partitions of 4.

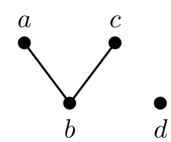
03/15

1. Find all linear extensions (see Example 16.9 of the text) of the 5 posets shown in Figure 16.3 from the text.

2. List all 4-element posets.

3. How many linear extensions do the posets below have?





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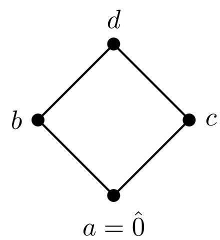
03/18

1. Consider the poset P shown below and the linear extension $L(a) = 1, L(b) = 3, L(c) = 2, L(d) = 4$ to answer the questions that follow.



   (a) Let $Z = Z_\zeta$ be the upper-triangular matrix associated with zeta function ζ_P of P. Find Z.

   (b) Use a CAS to find the matrix $M = M_\mu$ associated with the Möbius function μ_P of P.

   (c) Now let $\mu(x) = \mu(a, x)$ and compute $\mu(x)$ for all $x \in P$. Compare to the values that we obtained in class using the linear extension $K(a) = 1, K(b) = 2, K(c) = 3, K(d) = 4$.

2. Repeat the previous exercise for the divisor lattice D_{30}. If you are working with a classmate, choose different linear extensions and compare results.

03/20

1. For each of the following posets $(P, \leq)$, sketch the Hasse diagram and use Theorem 16.15 from the text to compute $\mu(x) := \mu(\hat{0}, x)$ for all $x \in P$.

   (a) $P = 2^{[4]}$ and the partial order is set containment. That is, $x \leq y$ if $x \subseteq y$.

   (b) $P = \Pi_4$, the (set) partition poset. Here the partial order is "refinement". That is, $x \leq y$ if each block in x is contained in a block in y. For example, $1/2/34 \leq 12/34$.

   (c) $P = D_{40}$, the divisor lattice with the usual partial order.

2. Construct the ζ matrix Z for the divisor lattice D_{40} and use a CAS to find the μ matrix M. Compare the first row of M with the values derived from the exercise above.

03/22 Read the proof of Theorem 7.6 in the text.

03/25

1. Use the Theorem 7.6 to re-prove Binomial inversion.

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2. Let P be the poset of the positive integers with $x \leq y \in P$ if $x \mid y$. Also, let $p_1, p_2, \ldots, p_k$ be k distinct primes and let $y = p_1 \cdot p_2 \cdots p_k$. Show that $[1, y]$ is isomorphic to B_k.

03/27

1. Let P and Q be posets. Show that $P \times Q$ with partial order as given by Definition 16.23 is a poset.

2. Construct a poset P such that $\mu(\hat{0}, x) = n$ for any $n \in \mathbb{Z}$.

3. Let P and Q be posets and consider the following alternative (partial) orders on $P \times Q$. Is $P \times Q$ a poset under the given order? *Note:* Throughout, we assume that $[p, p'] \subset P$ and $[q, q'] \subset Q$ and, for example, we write $p \leq p'$ instead of $p \leq_P p'$, etc.
   (a) $(p, q) \leq (p', q')$ if $p < p'$ or if $p = p'$ and $q \leq q'$.

   (b) $(p, q) \leq (p', q')$ if $p \leq p'$.

   (c) $(p, q) \leq (p', q')$ if $p < p'$ and $q < q'$ or $p = p'$ and $q = q'$.

03/29 The exercises below depend on the following results.

Proposition. Let $[x, y]$ be an interval in Π_n with the usual refinement (partial) order. If $y = B_1/B_2/\cdots/B_k$ and if each B_i splits into n_i blocks in x, then

$$[x, y] \cong \prod_{i=1}^{k} \Pi_{n_i} \tag{19}$$

In particular,

$$\mu(x, y) = \prod_{i=1}^{k} \mu_{\Pi_{n_i}}(\hat{0}, \hat{1}),$$

by Theorem 16.24. For example, let $x = 1/3/256/47$ and $y = 1347/256$ in Π_7. Then $x < y$ and

$$\begin{aligned} \mu(x, y) &= \mu_{\Pi_3}(\hat{0}, \hat{1})\mu_{\Pi_1}(\hat{0}, \hat{1}) \\ &= (-1)^2 2! \cdot (-1)^0 0! = 2 \end{aligned}$$

And the last line follows since

$$\mu(\Pi_n) := \mu_{\Pi_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)! \tag{20}$$

On Monday we will prove the above proposition and (20).

1. Use the above results to compute $\mu(x, \hat{1})$ for all $x \in \left\{ {4 \atop k} \right\}$ for $k \in [3]$. Also, compute $\mu(13/2/48/56/7, 123478/56)$ and $\mu(13/2/48/56/7, \hat{1})$ in Π_8.

2. Let $\{f_n\}_{n\geq 1}$ where $f_n = 2C_n - n$ and C_n are the Catalan numbers. Let Π_n be the set partition poset with the usual refinement order. On Quiz 8 we defined $F : \Pi_4 \to \mathbb{Z}$ by the rule $F(x) = f_{5-b(x)}$ where $b(x)$ is equal to the number of blocks in x. If we define $G(y) = \sum_{x\leq y} F(x)$, then by Möbius inversion

$$F(y) = \sum_{x\leq y} G(x)\mu(x,y) \tag{21}$$

Use (21) to show that $F(1234) = 24$.

04/01

1. Find an interval $[x, y] \subset \Pi_n$ such that
   (a) $\mu(x, y) = -12$

   (b) $\mu(x, y) = 96$

   Note: In each case, you will need to specify the value of n. Answers will not be unique.
2. Is there a positive integer n and an interval $[x, y]$ such that $\mu(x, y) = \pm 72$? Why or why not?

3. In our textbook's the definition of the *incidence algebra*, $I(P)$, it is stated that P must be a locally finite poset. Why is this?

4. Prove (19) in the proposition stated at the beginning of the assignments from 03/24.

04/03

1. Show that if $\{f(n)\}_{n\geq 1}$ is a multiplicative function, then so is

$$g(n) = \sum_{d|n} f(d)$$

2. Recall that Euler's function $\phi(n)$ counts the number of integers $1 \leq m \leq n$ such that m is relatively prime to n. Show by a counting argument that for $n \in \mathbb{P}$ one has

$$\sum_{d|n} \phi(d) = n$$

04/08

1. Let $\sigma(n) = \sum_{d|n} d$. That is, $\sigma(n)$ is the sum of the divisors of n.
   (a) Show that σ is multiplicative.

   (b) What does the Mobius Inversion formula say about σ?

2. Once again, let $\phi(n)$ be the Euler's totient function (see problem 2 from 04/03).

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(a) Show that $\phi(n) = n\sum_{d|n} \mu(d)/d$.

(b) Let p be prime and $k \in \mathbb{P}$. Show that $\phi(p^k) = p^k - p^{k-1}$.

(c) Let $\beta_1, \beta_2, \ldots, \beta_r$ be real numbers. Show that

$$\prod_{j=1}^{r}(1-\beta_j) = 1 - \sum_{i}\beta_i + \sum_{i<j}\beta_i\beta_j - \sum_{i<j<k}\beta_i\beta_j\beta_k + \cdots + (-1)^r\beta_1\beta_2\cdots\beta_r$$

(d) Use the Principle of Inclusion/Exclusion and part(c) above to prove that

$$\phi(n) = n\prod_{p|n}(1-p^{-1})$$

04/10

1. If f is multiplicative (and not identically 0) show that $f(1) = 1$.

2. Prove that for $n \in \mathbb{N}$ we have

$$\sum_{d|n}\mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

3. For $n \in \mathbb{N}$ define

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For example, $\Lambda(6) = \Lambda(10) = 0$ and $\Lambda(3) = \Lambda(27) = \log 3$.

(a) Show that

$$\log n = \sum_{d|n}\Lambda(d)$$

(b) Show that

$$\Lambda(n) = -\sum_{d|n}\mu(d)\log d$$

04/15

(a) Recall that the chromatic polynomial of the house H is $\chi(x) = \chi_H(x) = x(x-1)(x-2)(x^2-3x+3)$. Notice that $\chi(3) = 18$ so that there are 18 strictly compatible pairs (ρ, c). Here c is a proper 3-coloring of H and ρ is the induced orientation. Sketch 6 of the proper colorings using [3] and include the induced orientations, insuring that each of the 6 is acyclic.

(b) Do they same thing for barbell graph ($n = 3$). That is, find out how may strictly compatible pairs exist using [3], but this time sketch only 2 of the proper 3-colorings and include the induced orientations. Once again, insure that both orientations are acyclic.