Recall that we let S_n denote the set of permutations of n objects (usually taken to be $[n] = \{1, 2, 3, \ldots, n\}$) and that $|S_n| = n!$.

Theorem 1. Let $c_1, c_2, \ldots, c_n$ be nonnegative integers and suppose that $\sum_{j=1}^{n} j \cdot c_j = n$. Then the number of n-permutations with c_j cycles of length j, $j \in [n]$, is

$$\frac{n!}{c_1!c_2! \cdots c_n! \cdot 1^{c_1} 2^{c_2} \cdots n^{c_n}}$$

Proof: Let □

Definition. Let $n, k \geq 0$ and let $\begin{bmatrix} [n] \\ k \end{bmatrix} \subset S_n$ denote the permutations that contain exactly k cycles. We define the Stirling number of the first kind by the rule, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$ and for $n > 0$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \left| \begin{bmatrix} [n] \\ k \end{bmatrix} \right|$$

As usual, we define $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ whenever $k < 0$ or $k > n$.

For example, $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 11$ since there are precisely 11 permutations on 4 objects that have exactly 2 cycles. They are

(12)(34), (13)(24), (14)(23), (123)(4), (124)(3), (134)(2), (1)(234), (132)(4), (142)(3), (143

Note 1: There is an equivalent definition using the idea of the possible arrangements of n knights (in shining armor) seated at k round tables. Specifically, $\begin{bmatrix} n \\ k \end{bmatrix}$ counts the number of ways that n knights can sit around k identical circular tables, where no tables are allowed to remain empty. For a complete description, please see section 2.8.2 from the HHM text available here: `https://tinyurl.com/yd29ha7d`.

Note 2: Since neither the order of the cycles nor the (rotation) order within each cycle matters, it is traditional to list permutations in some sort of a *canonical* way. This is done by writing each cycle with its smallest element first and then listing cycles in increasing order by the first element of each. For example, the following two permutations are identical but the right-hand side is written in its canonical form.

$$(453)(16)(2) = (16)(2)(345)$$

We mention a few basic properties of these beautiful numbers.

Proposition 2. For $n \geq 1$,

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} = n! \tag{1}$$

Proof: Let $\pi \in S_n$. Then π has exactly zero cycles, or one cycle, or two cycles, etc. Now (1) follows from the Addition rule. □

As we might expect, these numbers also satisfy a recursion equation.

Proposition 3. For $n \geq k \geq 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} \tag{2}$$

Proof: **Question:** How many permutations of n objects have exactly k cycles?

Answer 1: By definition this is $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$.

Answer 2: There are $\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$ permutations where n is alone in a cycle. If n is not alone then we first create a k-cycle permutation on $n-1$ objects. There are $\left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]$ ways to do this. Now we may insert n to the right of each entry in any cycle. So there are $(n-1)\left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]$ cycles where n is not alone.

The result now follows by the Addition rule. □

n \ k	0	1	2	3	4	5	6	7	8	9	n!
0	1										1
1	0	1									1
2	0	1	1								2
3	0	2	3	1							6
4	0	6	11	6	1						24
5	0	24	50	35	10	1					120
6	0	120	274	225	85	15	1				720
7	0	720	1764	1624	735	175	21	1			5040
8	0	5040	13068	13132	6769	1960	322	28	1		40320
9	0	40320	109584	118124	67284	22449	4536	546	36	1	362880

Table 1: Stirling Type 1 Triangle

Table 1 shows the first 10 rows of Stirling's first triangle. The last column shows the sum of the row entries. By convention the omitted entries are zero. A quick inspection yields two obvious patterns. For $n > 0$,

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)! \tag{3}$$

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2} \tag{4}$$

We explore these and a few others in the exercises.

Proposition 4. For $n \geq 1$,

$$\sum_{k=1}^{n} k \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix} \tag{5}$$

Proof: The right-hand side corresponds to the number of two-cycle permutations on the set $\{0, 1, 2, \ldots, n\}$. By definition, this is $\left[{n+1 \atop 2}\right]$. For the left-hand side, consider a permutation with a single cycle identified in some way. For example, let $\pi = (134)(25)(687)(9) \in S_9$ and suppose that we identify the 2nd cycle with an underline, say $(134)\underline{(25)}(687)(9)$. Clearly, we can do this a total of 3 more times. Now since we can do this k times for each permutation with k cycles, the total number of permutations identified in this way is $\sum_{k=1}^{n} k \left[{n \atop k}\right]$. Denote the set of *identified* cycles by the symbol $\mathfrak{C}_n$. We illustrate a one-to-one correspondence between $\mathfrak{C}_n$ and $\begin{bmatrix} [n+1] \\ 2 \end{bmatrix}$ below.

Consider the following:

$$(134)\underline{(25)}(687)(9) \implies (9)(687)(134)\underline{(25)} \implies (0)(9)(687)(134)\underline{(25)} \implies (09687134)(25) \tag{6}$$

Here's another illustration.

$$\underline{(134)}(25)(687)(9) \implies (9)(687)(25)\underline{(134)} \implies (0)(9)(687)(25)\underline{(134)} \implies (0968725)(134)$$

Now reverse the procedure. Say we start with the right-hand side of (6).

$$(09687134)(25) \implies (9687134)\underline{(25)} \implies (9)(687)(134)\underline{(25)} \implies (134)\underline{(25)}(687)(9)$$

which is the left-hand side of (6).

The map from $\mathfrak{C}_n$ to $\begin{bmatrix}[n+1]\\ 2\end{bmatrix}$ is clear. What about its inverse? So let $(0a_1a_2\cdots a_{n-k})(b_1b_2\cdots b_k) \in \begin{bmatrix}[n+1]\\ 2\end{bmatrix}$. We construct an *identified* permutation

$$\cdots(c_3)(c_2)(c_1) \in \mathfrak{C}_n$$

as follows. Starting with a_1 we proceed to a_2, a_3, ... until we encounter an entry $a_j < a_1$ and we let $(c_1) = (a_1a_2\cdots a_{j-1})$. Now we repeat the process, this time starting with $a_j, a_{j+1}, \cdots$ until we reach an entry $a_{j+m} < a_j$. Now let $(c_2) = (a_ja_{j+1}\cdots a_{j+m-1})$. This procedure will terminate eventually. Finally, we insert the identified cycle $\underline{(b_1b_2\cdots b_k)}$ into its proper position. □

Remark: What purpose does the introduction of "0" serve?

Now let's take a look at the generating functions for these numbers. For $n \geq 0$ let $g_n(x) = \sum_k \begin{bmatrix}n\\k\end{bmatrix} x^k$. Since $\begin{bmatrix}0\\k\end{bmatrix} = 1$ for $k = 0$ and zero otherwise, we have $g_0(x) = 1$.

So let $n > 0$. Then by (2) we have

$$\begin{aligned} g_n(x) &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k \\ &= x \sum_k \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^k + (n-1) \sum_k \begin{bmatrix} n-1 \\ k \end{bmatrix} x^k \\ &= x g_{n-1}(x) + (n-1) g_{n-1}(x) \\ &= (x+n-1) g_{n-1}(x) \\ &\vdots \\ &= x(x+1)(x+2)\cdots(x+n-1) \end{aligned}$$

You may recall that we have already encountered similar identities before. Define the rising factorial, $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$, then

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k = x^{\overline{n}} \tag{7}$$

Also, see Exercise 8.

We should also mention that Stirling numbers of the first kind have a "signed" counterpart. More specifically, we have

$$s(n,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \tag{8}$$

Note: Many authors refer to $\begin{bmatrix} n \\ k \end{bmatrix}$ as the *signless* Stirling numbers of the first kind.

Exercises

1. Prove identities (3) and (4).

2. Show that for $n \geq 2$,

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k = 0$$

   Hint: If π is a permutation, then 2 must appear in the first or second cycle. Now devise an involution that *increases* by one the parity on the number of cycles of π if 2 appears in its first cycle and *decreases* the parity otherwise.

3. Let $M(n) = (m_{jk})$ be an $n \times n$ matrix with $m_{jk} = \left[{j \atop k}\right]$. For example,

$$M(5) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 6 & 11 & 6 & 1 \end{pmatrix}$$

   Notice that $M(5)$ is invertible. Find it's inverse and postulate the general form of $M(n)^{-1}$.

4. Use a combinatorial argument to show that

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = \frac{n!}{2} \sum_{m=1}^{n-1} \frac{1}{m(n-m)}$$

5. Use a combinatorial argument to find a simple formula for $\begin{bmatrix} n \\ n-2 \end{bmatrix}$.

6. Prove that if $n, m \geq 0$ then

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}$$

   Hint: Try induction on n. For a combinatorial approach, modify the *identified-cycle* argument used in the proof of Proposition 4.

7. Prove that if $n \geq 0$ then

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} 2^k = (n+1)!$$

   Hint: The previous exercise should help.

8. Prove that if $n \geq 0$ then

$$\sum_k (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k = x^{\underline{n}} \tag{9}$$

9. Use a combinatorial argument to prove that for $n, m \geq 0$

$$\sum_{k=0}^{n} \begin{bmatrix} n-k \\ m-1 \end{bmatrix} n^{\underline{k}} = \begin{bmatrix} n+1 \\ m \end{bmatrix} \tag{10}$$