THROUGHOUT THIS EXAM, THE INSTRUCTION "FIND THE GENERATING FUNCTION" ALWAYS MEANS FIND THE CLOSED FORM OF THE GENERATING FUNCTION.

1. Let $\mathcal{B} = \{\bullet, \bullet\ \bullet{-}\bullet, \bullet{-}\bullet{-}\bullet\}$. So $\mathcal{B}$ has 1 object of size one and 2 objects of size three. The first few terms in the counting sequence for the class $\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$ are $1, 1, 1, 3, 5, 7, 13, 23, \ldots$

   Answer the questions below. *Note:* To be clear, $\bullet{-}\bullet \notin \mathcal{A}$.

   (a) (6 points) Notice that $(\bullet, \bullet\ \bullet{-}\bullet, \bullet) \in \mathcal{A}$. Now list the other 6 elements of size five in $\mathcal{A}$.

   Solution:

   The 6 other elements are

$$(\bullet, \bullet, \bullet, \bullet, \bullet)$$
$$(\bullet\ \bullet{-}\bullet, \bullet, \bullet),\ (\bullet, \bullet, \bullet\ \bullet{-}\bullet)$$
$$(\bullet{-}\bullet{-}\bullet, \bullet, \bullet),\ (\bullet, \bullet{-}\bullet{-}\bullet, \bullet),\ (\bullet, \bullet, \bullet{-}\bullet{-}\bullet)$$

   (b) (7 points) Find the generating function of $\mathcal{A}$.

   Solution:

$$A(x) = \frac{1}{1 - x - 2x^3}$$

   (c) (7 points) Find the generating function of $\mathcal{C} = \mathrm{SEQ}(\bullet{-}\bullet{-}\bullet\,\mathcal{A})$.

   Solution:

$$C(x) = \frac{1}{1 - x^3 A(x)} = \frac{1}{1 - \frac{x^3}{1-x-2x^3}} = \frac{1 - x - 2x^3}{1 - x - 3x^3}$$

2. (14 points) Let d_n count the number of derangements in $\mathfrak{S}_n$. These numbers satisfy the following recursion

$$d_{n+1} = (n+1)d_n + (-1)^{n+1}, \quad n \geq 0,\ d_0 = 1 \tag{1}$$

Note: The first few derangement numbers are 1, 0, 1, 2, 9, 44, 265, 1854, 14833.

(a) Give a combinatorial proof of the recursion in (1).

(b) Use (1) and the Wilf rules to (re-)derive the following result.

$$\sum_n d_n \frac{x^n}{n!} = \frac{e^{-x}}{1-x} \tag{2}$$

Solution:

Let $g(x) = \sum_n d_n \dfrac{x^n}{n!}$. Then (1) together with the Wilf rules implies

$$g'(x) = xg'(x) + g(x) - e^{-x}$$

Rearranging produces

$$g'(x)(1-x) - g(x) = -e^{-x}$$

or

$$D((1-x)g(x)) = -e^{-x}$$

Integrating both sides yields

$$\begin{aligned}(1-x)g(x) &= e^{-x} + C \\ &= e^{-x} + 0\end{aligned}$$

which is (2).

3. (20 points) Let $\mathcal{S}(\cdot) = \binom{\cdot}{1}$. Answer the questions below.

(a) Find the exponential generating function $F_{\mathcal{S}}(x) = \sum_{n\geq 0} \binom{n}{1} x^n/n!$.

Solution:

This one is straightforward.

$$F_{\mathcal{S}}(x) = \sum_{n\geq 0} n\,\frac{x^n}{n!} = xD(e^x) = xe^x$$

(b) List the distinct elements in $(\mathcal{S} \times \mathcal{S})([3])$. *Note:* These elements should be written as ordered pairs.

Solution:

$$(1,2),\ (1,3),\ (2,1),\ (2,3),\ (3,1),\ (3,2)$$

(c) Find the exponential generating function $F_{\mathcal{S}\times\mathcal{S}}(x)$. *Note:* $F_{\mathcal{S}\times\mathcal{S}}(x) \neq F_{\binom{\cdot}{2}}(x)$.

Solution:

By the Product Rule,

$$F_{\mathcal{S}\times\mathcal{S}}(x) = x^2 e^{2x}$$

(d) Use the exponential formula to find the exponential generating function for the *partition structure* $\Pi(\mathcal{S})$. In other words, find $F_{\Pi(\mathcal{S})}(x)$. *Note:* $\mathcal{S} = \overline{\mathcal{S}}$

Solution:

By the exponential formula

$$F_{\Pi(\mathcal{S})}(x) = e^{F_{\overline{\mathcal{S}}}(x)} = e^{xe^x}$$

The first few terms of the counting sequence are

$$1, 1, 3, 10, 41, 196, 1057, 6322, 41393, 293608, 2237921, 18210094$$

4. (16 points) Let n and k be integers. Let $\left\lfloor {[n] \atop k} \right\rfloor$ be the collection of all partitions of $[n]$ into k linearly ordered blocks. As usual, let $\left\lfloor {0 \atop 0} \right\rfloor = 1$ and for $n > 0$, let $\left\lfloor {n \atop k} \right\rfloor = \left| \left\lfloor {[n] \atop k} \right\rfloor \right|$. For example, $\left\lfloor {[3] \atop 2} \right\rfloor = \{12/3,\ 21/3,\ 13/2,\ 31/2,\ 23/1,\ 32/1\}$. It follows that $\left\lfloor {3 \atop 2} \right\rfloor = 6$. Notice that only the ordering within each block matters, not the order of the blocks themselves, so $32/1 = 1/32$, etc. It turns out that these numbers satisfy the following recursion.

$$\left\lfloor {n+1 \atop k} \right\rfloor = (n+k)\left\lfloor {n \atop k} \right\rfloor + \left\lfloor {n \atop k-1} \right\rfloor \tag{3}$$

together with additional boundary conditions $\left\lfloor {n \atop k} \right\rfloor = 0$ whenever $n < 0$ or $k \leq 0$ or $k > n$.

(a) Find a combinatorial proof of the recursion (3).

Solution:

The left-hand side counts the number of partitions of $[n+1]$ into k linearly ordered blocks. Throughout the remainder of this proof, a partition means a partition with linearly ordered blocks.

Now for any partition in $\left\lfloor {[n+1] \atop k} \right\rfloor$, $n+1$ is either alone in a block or it is not.
In the first case, we can append $n+1$ to any of the partitions in $\left\lfloor {[n] \atop k-1} \right\rfloor$ to create a partition in $\left\lfloor {[n+1] \atop k} \right\rfloor$. Clearly there are $\left\lfloor {n \atop k-1} \right\rfloor$ ways to do this.

Otherwise, we can choose $\lambda \in \left\lfloor {[n] \atop k} \right\rfloor$, say $\lambda = B_1/B_2/\cdots/B_k$. Now we can place $n+1$ at the beginning of any block, e.g., $\lambda^j = B_1/B_2/\cdots/(n+1)B_j/\cdots/B_k$, so there are $k\left\lfloor {n \atop k} \right\rfloor$ ways to do this. Or we can place $n+1$ after any element (within any block), so there must be $n\left\lfloor {n \atop k} \right\rfloor$ ways to do this.

Since the 3 cases are distinct, we have shown that

$$\left\lfloor {n+1 \atop k} \right\rfloor = \left\lfloor {n \atop k-1} \right\rfloor + k\left\lfloor {n \atop k} \right\rfloor + n\left\lfloor {n \atop k} \right\rfloor$$

which is (3).

(b) Let $L_k(x) = \sum_{n \geq 0} \left[{n \atop k}\right] \frac{x^n}{n!}$. It turns out that

(4) $$L_k(x) = \frac{1}{k!} \left(\frac{x}{1-x} \right)^k$$

Verify (4) when $k = 1$.

Solution:

$$\begin{aligned} L_1'(x) &= \sum_n \left[{n+1 \atop 1}\right] \frac{x^n}{n!} \\ &= \sum_n \left[{n \atop 0}\right] \frac{x^n}{n!} + 1 \sum_n \left[{n \atop 1}\right] \frac{x^n}{n!} + \sum_n n \left[{n \atop 1}\right] \frac{x^n}{n!} \\ &= L_0(x) + L_1(x) + xL_1'(x) \end{aligned}$$

Rearranging yields

$$L_1'(x)(1-x) - L_1(x) = L_0(x) = 1$$

or

$$D\left((1-x)L_1(x)\right) = 1$$

Integrating both sides produces

$$\begin{aligned} (1-x)L_1(x) &= x + C \\ &= x + 0 \end{aligned}$$

It follows that

$$L_1(x) = \frac{x}{1-x}$$

as expected.

5. (10 points) Let $\mathcal{S}(\cdot) = \binom{\cdot}{1}$. We saw in problem 3 above that $F_{\Pi(\mathcal{S})}(x) = e^{xe^x}$. Let

$$i_n = n![x^n]F_{\Pi(\mathcal{S})}(x) = n![x^n]e^{xe^x}$$

Find a sum formula for i_n. *Note:* The right-hand side of (6) is an example of a sum formula.

Solution:

$$\begin{aligned}
\frac{i_n}{n!} &= [x^n]e^{xe^x} \\
&= [x^n]\sum_m x^m \frac{e^{mx}}{m!} \\
&= \sum_m \frac{1}{m!}[x^{n-m}]e^{mx} \\
&= \sum_m \frac{1}{m!}[x^{n-m}]\sum_k m^k \frac{x^k}{k!} \\
&= \sum_m \frac{1}{m!}\frac{m^{n-m}}{(n-m)!}
\end{aligned}$$

It follows that

$$\begin{aligned}
i_n &= \sum_m \frac{n!}{m!(n-m)!} m^{n-m} \\
&= \sum_m \binom{n}{m} m^{n-m}
\end{aligned}$$

It is worth mentioning that $i_0 = 1$ since $F_{\Pi(\mathcal{S})}(0) = 1$, but the sum formula above returns the indeterminate expression 0^0. In this case, we should specify that we define $0^0 = 1$.

6. (10 points) Let $l_0 = 1$ and for $n > 0$ let l_n count the number of ways to partition $[n]$ into an arbitrary number of nonempty linearly ordered blocks. In other words,

$$l_n = \sum_k \left\lfloor {n \atop k} \right\rfloor, \quad n \geq 0$$

Prove that the closed form of the exponential generating function $L(x) = \sum_n l_n\, x^n/n!$ is

$$L(x) = e^{\frac{x}{1-x}} \tag{5}$$

Note: For this problem you may freely use my posted lecture notes or any of the other references listed on my Math 482 pages, but please do not use OEIS.

Solution:

Let $\mathcal{S}(\cdot) = \left\lfloor {\cdot \atop 1} \right\rfloor$. As we saw in problem 4 above,

$$F_{\mathcal{S}}(x) = L_1(x) = \frac{x}{1-x} = F_{\overline{\mathcal{S}}}(x)$$

Notice that the last equality holds since $\overline{\mathcal{S}} = \mathcal{S}$.

So the problem is asking us to partition $[n]$ in all possible ways and to linearly order each block in all possible ways. In other words, this problem is describing the partition structure $\Pi(\mathcal{S})$. Thus

$$L(x) = \sum_n l_n \frac{x^n}{n!} = F_{\Pi(S)}(x)$$

It now follows by the exponential formula that

$$F_{\Pi(S)}(x) = e^{F_{\overline{\mathcal{S}}}(x)} = e^{\frac{x}{1-x}}$$

7. (10 points) Work only one of the parts below. **Cross out the part that you do not want graded.** I will award zero points if you fail to cross out one of the parts.

(a) Prove that for $n \geq 1$, we have

$$c_n := \begin{bmatrix} n+1 \\ 2 \end{bmatrix} = n! \sum_{k=1}^{n} \frac{1}{k} \tag{6}$$

Solution:

Let $C(x) = \sum_n c_n \, x^n/n!$. It is routine (using either the Wilf Rules or the recursion for cycle numbers) to show that

$$C(x) = \frac{1}{1-x} \ln \frac{1}{1-x}$$

It follows that

$$\begin{aligned} \frac{1}{n!} \begin{bmatrix} n+1 \\ 2 \end{bmatrix} = [x^n]C(x) &= [x^n] \frac{1}{1-x} \ln \frac{1}{1-x} \\ &= [x^n] \frac{1}{1-x} \sum_{n \geq 1} \frac{x^n}{n} \\ &\overset{(*)}{=} [x^n] \sum_{n \geq 1} \sum_{k=1}^{n} \frac{1}{k} \, x^n \\ &= \sum_{k=1}^{n} \frac{1}{k} \end{aligned}$$

Here (*) follows by Wilf Rule 5. Multiplying through by $n!$ yields (6).

(b) Let F_n be the shifted Fibonacci numbers: $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$ and let d_n be the derangement numbers described in problem 2. Express the sum formula below in a simple closed form.

$$s_n = \sum_k \binom{n}{k} (kF_{k-1} - F_k) d_{n-k} \tag{7}$$

Note: You may be able to guess the closed form, but you must justify your claim to receive any credit.