Formal Power Series

Definition 1. A **formal power series** is an expression of the form

$$a_0 + a_1x + a_2x^2 + \cdots = \sum_{n\geq 0} a_nx^n$$

The sequence $\{a_n\}_{n\geq 0}$ is called the **sequence of coefficients**. Two series are equal if they have the same sequence of coefficients.

Now let $\mathbb{C}[[x]]$ denote the space of all formal power series. That is, let

$$\mathbb{C}[[x]] = \left\{ \sum_{n\geq 0} a_nx^n \,|\, a_n \in \mathbb{C} \text{ for } n \geq 0 \right\}$$

It turns out that $\mathbb{C}[[x]]$ is an algebra, the *algebra of formal power series*, under the operations of addition, scalar multiplication, and multiplication defined below.

$$\sum_{n\geq 0} a_nx^n + \sum_{n\geq 0} b_nx^n = \sum_{n\geq 0} (a_n + b_n)x^n, \tag{1}$$

$$c\sum_{n\geq 0} a_nx^n = \sum_{n\geq 0} ca_nx^n, \qquad c \in \mathbb{C} \tag{2}$$

$$\sum_{n\geq 0} a_nx^n \sum_{n\geq 0} b_nx^n = \sum_{n\geq 0} c_nx^n, \qquad \text{where } c_n = \sum_{k=0}^{n} a_kb_{n-k} \tag{3}$$

We observe that these series do *converge* at $x = 0$. In particular, if $f(x) \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0}$ then $f(0) = a_0$. In general however, we do not concern ourselves with notions of convergence (of real or complex numbers), hence the term *formal*. (*Note:* However, we will introduce an elementary notion of convergence within $\mathbb{C}[[x]]$. See Definition 5 below.)

Formal power series are primarily a means of studying the sequences $\{a_n\}_{n\geq 0}$, called the *sequence of coefficients*. To that end we introduce the *linear functional*, $[x^n]$

$$[x^n] : \mathbb{C}[[x]] \to \mathbb{C}$$

defined by the rule

$$[x^n]\sum_{n\geq 0} a_n x^n = a_n$$

for any integer $n \geq 0$. The operator is clearly linear since by (1) and (2) we have

$$\begin{aligned}[x^n]\left(\sum_{n\geq 0} a_n x^n + \sum_{n\geq 0} b_n x^n\right) &= [x^n]\sum_{n\geq 0}(a_n + b_n)x^n \\ &= a_n + b_n = [x^n]\sum_{n\geq 0} a_n x^n + [x^n]\sum_{n\geq 0} b_n x^n\end{aligned}$$

and

$$\begin{aligned}[x^n]c\sum_{n\geq 0} a_n x^n &= [x^n]\sum_{n\geq 0} c a_n x^n \\ &= ca_n = c[x^n]\sum_{n\geq 0} a_n x^n\end{aligned}$$

We look at some examples of the usage of this operator below.

Now consider the following product (of formal power series).

$$\begin{aligned}(1-x)\sum_{n\geq 0} x^n &= \sum_{n\geq 0} x^n - \sum_{n\geq 0} x^{n+1} \\ &= 1 + \sum_{n\geq 1} x^n - \sum_{n\geq 0} x^{n+1} \\ &= 1 + \sum_{n\geq 0} x^{n+1} - \sum_{n\geq 0} x^{n+1} \\ &= 1\end{aligned}$$

Because of this result, it seems reasonable to say that $\sum_n x^n$ and $1-x$ are reciprocals. We have the following proposition.

Proposition 2. A formal power series $f(x) = \sum_{n\geq 0} a_n x^n$ has a reciprocal, which we will denote by $1/f(x)$, if and only if $a_0 \neq 0$. When it exists, $1/f(x)$ is unique.

For a proof, see the Wilf text.

We also need to discuss the *inverse* of a formal power series. That is, if f is a formal power series, under what conditions does there exists another (formal) power series such that

$$f(g(x)) = g(f(x)) = x \tag{4}$$

Now suppose that $f(x) = \sum_{n\geq 0} f_n x^n$ and $g(x) = \sum_{n\geq 0} g_n x^n$. It seems reasonable to define

$$f(g(x)) = \sum_{n\geq 0} f_n(g(x))^n =^? \sum_{n\geq 0} b_n x^n$$

But what are the b_n's? Specifically, what is b_0? It doesn't take too long to recognize that

$$b_0 = f_0 + f_1 g_0 + f_2 g_0^2 + \cdots$$

That is, to determine that value of b_0 requires an *infinite* process, unless f happens to be a polynomial. On the other hand, if $g_0 = 0$ then each of the coefficients b_n can be computed in a finite number of steps. We make the following definition.

Definition 3. $f(g(x))$ is defined if and only if f is a polynomial or $g_0 = 0$.

Proposition 4. Let f and g be formal power series that satisfy (4) and let $a_1 = [x^1]f(x)$ and $b_1 = [x^1]g(x)$. Then $a_1 \neq 0$ and $b_1 \neq 0$.

For a proof see the Wilf text.

It turns out that one can introduce a topology on $\mathbb{C}[[x]]$. However, such a discussion will take us too far away from our main focus. Instead, it will suffice to introduce an elementary notion of *convergence* in $\mathbb{C}[[x]]$.

Definition 5. Let $\{f_n(x)\}_{n\geq 0}$ be a sequence of formal power series in $\mathbb{C}[[x]]$. We say that $f_n(x)$ **converges** to $f(x) = \sum_n a_n x^n \in \mathbb{C}[[x]]$ if the sequence of coefficients

$$[x^k]f_1(x),\ [x^k]f_2(x),\ [x^k]f_3(x), \ldots$$

eventually stabilizes to the value a_k, and that this result holds for all k. In this case, we write

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{or} \quad f_n(x) \to f(x) \text{ as } n \to \infty \tag{5}$$

Otherwise we say that the sequence **diverges** or the limit does not exist. *Note:* We often omit the symbol $n \to \infty$ when using the latter notation.

As a trivial but important example, consider the formal power series $A(x) = \sum_{n\geq 0} a_n x^n$. Now define the (partial sum) sequence $s_k(x) = \sum_{j=0}^{k} a_j x^j$. Then $s_k(x) \to A(x)$ since for all n

$$\begin{aligned}[x^n]s_0(x),\ [x^n]s_1(x),\ [x^n]s_2(x), \ldots,\ [x^n]s_n(x),\ [x^n]s_{n+1}(x),\ [x^n]s_{n+2}(x), \ldots \\ = 0,\ 0,\ 0, \ldots,\ a_n,\ a_n,\ a_n, \ldots\end{aligned}$$

as expected.

There is an equivalent notion of convergence. Let $f(x) = \sum_n a_n x^n \in \mathbb{C}[[x]]$ and define the **degree** of $f(x)$, written as $\deg(f(x))$, to be the smallest n such that $a_n \neq 0$. If $f(x) = 0$ define $\deg(f(x)) = \infty$. We leave it as an exercise to state and prove an equivalent definition of convergence using degree (see Exercise 3).

Theorem 6. (Sagan) Let $\{f_k\}_{k\geq 0}$ be a sequence of formal power series. Then $\sum_{k\geq 0} f_k(x)$ exists (as a formal power series) if and only if $\lim_{k\to\infty} \deg(f_k(x)) = \infty$.

Proof: Suppose that $\{f_k(x)\}_{k\geq 0} \subset \mathbb{C}[[x]]$ and let $s_k(x) = \sum_{j=0}^{k} f_j(x)$. Since $\sum_{k\geq 0} f_k(x)$ exists, there is an integer $K = K(n)$ such that $[x^n]s_K(x),\ [x^n]s_{K+1}(x),\ [x^n]s_{K+2}(x), \ldots$ is constant. In other words, for all positive m

$$\begin{aligned}0 &= [x^n]s_{K+m}(x) - [x^n]s_K(x) \\ &= [x^n] \sum_{j=K+1}^{K+m} f_j(x)\end{aligned}$$

Thus, $\deg(f_k(x)) > n$ for all $k \geq K$. Now let $n \to \infty$ and the result follows. We leave the reverse implication as an exercise. $\square$

We are now in a position to give a more formal treatment of the composition of two formal power series (see Definition (3)).

Corollary 7. Let $f(x), g(x) \in \mathbb{C}[[x]]$. Then $f(g(x)) \in \mathbb{C}[[x]]$ if and only if $f(x)$ is a polynomial or $g(0) = 0$.

Proof: We leave the forward direction as an exercise. For the reverse implication, if $f(x)$ is a polynomial then the result follows because $\mathbb{C}[[x]]$ is an algebra under the algebraic operations defined in (1)-(3). On the other hand, suppose that $f(x) = \sum_{n\geq 0} f_n x^n$ and $g(0) = 0$. Now let $F_n(x) = f_n(g(x))^n$ and observe that $\deg(F_n(x)) \geq n$. So by the previous theorem,

□

$$f(g(x)) = \sum_{n\geq 0} f_n(g(x))^n = \sum_{n\geq 0} F_n(x) \in \mathbb{C}[[x]]$$

Theorem 8. Let $\{f_k(x)\}_{k\geq 0} \subset \mathbb{C}[[x]]$ and suppose that $f_k(0) = 0$ for each k. Then $\prod_{k\geq 0}(1 + f_k(x)) \in \mathbb{C}[[x]]$ if and only if $\lim_{k\to\infty} \deg(f_k(x)) = \infty$.

As an immediate consequence we have

Corollary 9. $\prod_{k\geq 1}(1 + x^k) \in \mathbb{C}[[x]]$

Calculus of Formal Power Series

Once again let $f(x) = \sum_{n\geq 0} a_n x^n$. Then we define $f'(x)$ to be the formal power series $f'(x) = \sum_{n\geq 1} n a_n x^{n-1}$. It turns out that the usual derivative rules such as the sum, product, and quotient rules hold for formal power series. The interested reader is encouraged to explore the Wilf text.

We conclude this section with several examples of extractionator usage.

Example 10. Let $A(x) \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0}$ and let k be a positive integer.

a. Let $g(x) = 2 + 3x^2 - x^7$. Then $[x^n]g(x) = 0$ except

$$[x^0]g(x) = 2,\ [x^2]g(x) = 3,\ [x^7]g(x) = -1$$

b. Let $G(x) = \sum_{n\geq 0} g_n x^{2n+1}$. Then

$$[x^{2n}]G(x) = 0$$

and

$$[x^{2n+1}]G(x) = g_n$$

c. $[x^n]\dfrac{1}{1-x} = 1$ since

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n\geq 0} x^n$$

d. $[x^n]\dfrac{1}{1+x} = (-1)^n$ since

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n\geq 0} (-1)^n x^n$$

e. $[x^n]x^k A(x) = [x^{n-k}]A(x) = a_{n-k}$ since

$$\begin{aligned} x^k A(x) &= x^k(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \\ &= a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + a_3 x^{k+3} \cdots \end{aligned}$$

Notice that the coefficient subscripts are k units **less** than the corresponding exponent, as we claimed.
By a similar argument

$$[x^n]\frac{1}{x^k}A(x) = [x^{n+k}]A(x) = a_{n+k}$$

f. $[x^n]\dfrac{A(x)}{1-x} = \displaystyle\sum_{k=0}^{n} a_k$ since

$$\frac{A(x)}{1-x} = \sum_{n\geq 0} \sum_{k=0}^{n} a_k x^n$$

For example, let $f(x) = (1 - x - x^2)^{-1}$. Then

$$\begin{aligned} \frac{f(x)}{1-x} &= \frac{1}{(1-x)(1-x-x^2)} \\ &= 1 + (1+1)x + (1+1+2)x^2 + (1+1+2+3)x^3 + (1+1+2+3+5)x^4 + \cdots \\ &= 1 + 2x + 4x^2 + 7x^3 + 12x^4 + 20x^5 + 33x^6 + 54x^7 + 88x^8 \end{aligned}$$

so that $[x^6]f(x)(1-x)^{-1} = 1 + 1 + 2 + 3 + 5 + 8 + 13 = 33$.

Example 11.

a.

$$\begin{aligned}
[x^n]\frac{2x}{(1-x)^8} &= 2[x^n]\frac{x^6}{x^6}\,\frac{x}{(1-x)^8} \\
&= 2[x^{n+6}]\frac{x^7}{(1-x)^8} \\
&= 2[x^{n+6}]\sum_m \binom{m}{7} x^m \\
&= 2\binom{n+6}{7}
\end{aligned}$$

b. Let $G(x) = \dfrac{x}{(1+2x)^4}$. Then

$$\begin{aligned}
[x^n]G(x) &= [x^n]\frac{(-2x)^2}{(-2x)^2}\,\frac{-2}{-2}\,\frac{x}{(1-(-2x))^4} \\
&= [x^{n+2}]\frac{-1}{8}\,\frac{(-2x)^3}{(1-(-2x))^4} \\
&= \frac{-1}{8}[x^{n+2}]\sum_m \binom{m}{3}(-2x)^m \\
&= \frac{-1}{8}[x^{n+2}]\sum_m \binom{m}{3}(-2)^m x^m \\
&= \frac{-1}{8}\binom{n+2}{3}(-2)^{n+2}
\end{aligned}$$

The first 9 terms of this sequence are

$$0,1,-8,40,-160,560,-1792,5376,-15360$$

and the first 9 terms of the Taylor's expansion of $G(x)$ are

$$x - 8x^2 + 40x^3 - 160x^4 + 560x^5 - 1792x^6 + 5376x^7 - 15360x^8$$

as we would expect.

Exercises

1. Prove the reverse implication in Theorem 6.

2. Prove the forward direction of Corollary 7. That is, suppose that $f(x), g(x) \in \mathbb{C}[[x]]$. If $f(g(x)) \in \mathbb{C}[[x]]$ then $f(x)$ is a polynomial or $g(0) = 0$.
   Hint: If $f(x)$ is not a polynomial **and** $g(0) \neq 0$, show that $f(g(x)) \notin \mathbb{C}[[x]]$.

3. State a definition of convergence in $\mathbb{C}[[x]]$ in terms of the *degree* operator. Show that this definition is equivalent to the one given in Definition 5.

4. Let $\{b_n\}_{n\geq 1}$ be a sequence of nonnegative integers. Prove that $\prod_{n\geq 1}(1+x^n)^{b_n} \in \mathbb{C}[[x]]$.