Basic Definitions and Introductory Examples

We motivate the definitions that are to follow by revisiting a few examples that we have previously encountered. Each of these introductory examples will include a finite collection of objects called a *class*. The size of each object in the class can be expressed as a nonnegative integer.

Definition 1. A **combinatorial class** (or class) is a finite or countable set of objects with a size function defined for each object in the set and such that

(i) the size of each object is a nonnegative integer

(ii) the number of objects of any give size is finite

If $\mathcal{A}$ is such a class, then the size of $a \in \mathcal{A}$ is denoted $|a|$. Our goal is to encode this information into a symbolic form called an ordinary *generating function*

$$\sum_{a\in\mathcal{A}} x^{|a|} \tag{1}$$

Also, let $\mathcal{A}_n \subset \mathcal{A}$ be the subset of all objects in $\mathcal{A}$ of size n and let $A_n = |\mathcal{A}_n|$. We call $\{A_n\}_{n\geq 0}$ the *counting sequence* (or coefficient sequence) of $\mathcal{A}$. Now we can rewrite the *combinatorial* form defined in (1) into the *algebraic* form described below.

$$\sum_{n\geq 0} A_n x^n \tag{2}$$

Example 2. Consider permutations in $\mathfrak{S}_5$ and suppose that the size of each permutation is defined by its *inversion number*. What is the generating function (polynomial) of the following class?

$\mathcal{B} = \{\pi_1 = 2\ 1\ 3\ 4\ 5,\ \pi_2 = 2\ 1\ 3\ 5\ 4,\ \pi_3 = 1\ 2\ 4\ 5\ 3,\ \pi_4 = 1\ 4\ 5\ 2\ 3,\ \pi_5 = 4\ 1\ 3\ 2\ 5,\ \pi_6 = 5\ 1\ 2\ 4\ 3\}$

Observe that the corresponding inversion tables are

$$I(\pi_1) = 10000,\ I(\pi_2) = 10010,\ I(\pi_3) = 00200,\ I(\pi_4) = 02200,\ I(\pi_5) = 12100,\ I(\pi_6) = 11210$$

So that

$$\operatorname{inv}\pi_1 = 1,\ \operatorname{inv}\pi_2 = 2,\ \operatorname{inv}\pi_3 = 2,\ \operatorname{inv}\pi_4 = 4,\ \operatorname{inv}\pi_5 = 4,\ \operatorname{inv}\pi_6 = 5$$

It follows that

$$B(x) = \sum_{\pi\in\mathcal{B}} x^{|\pi|} = \sum_{\pi\in\mathcal{B}} x^{\operatorname{inv}\pi} \tag{3}$$

$$= \sum_{j=1}^{6} x^{\operatorname{inv}\pi_j} = x^1 + x^2 + x^2 + x^4 + x^4 + x^5$$

$$= x + 2x^2 + 2x^4 + x^5 \tag{4}$$

Notice that the counting sequence of $\mathcal{B}$ is $\{0, 1, 2, 0, 2, 1\}$ and we have expressed our generating polynomial in its combinatorial form in (3) and its algebraic form in (4).

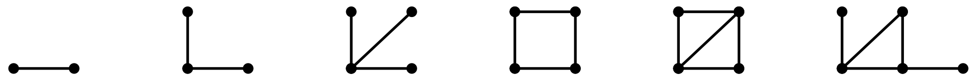

Figure 1: A finite family $\mathcal{G}$ of graphs

Example 3. Consider the family $\mathcal{C}$ of connected graphs shown in Figure 1. Suppose that we define the size each of each graph by the number of vertices. Then the associated generating function is

$$C(x) = \sum_{g \in \mathcal{G}} x^{\operatorname{ord} g} = x^2 + x^3 + 3x^4 + x^5$$

If instead we use the number of edges to denote size, then we obtain

$$H(y) = \sum_{g \in \mathcal{G}} y^{\operatorname{size} g} = y + y^2 + y^3 + y^4 + 2y^5$$

Using both vertices and edges yields

$$K(x, y) = x^2 y + x^3 y^2 + x^4 y^3 + x^4 y^4 + x^4 y^5 + x^5 y^5$$

Example 4. Let $\mathcal{P}$ be the set of permutations. That is, $\mathcal{P} = \bigcup_{n \geq 0} \mathfrak{S}_n$. Now for $\pi \in \mathfrak{S}_n \subset \mathcal{P}$, let $|\pi| = n$. Then the counting sequence $P_n = |\mathfrak{S}_n| = n!$.

Definition 5. Two classes $\mathcal{A}$ and $\mathcal{B}$ are said to be combinatorially equivalent if and only if their counting sequences are identical. In this case we write $\mathcal{A} \cong \mathcal{B}$. Equivalently, $\mathcal{A} \cong \mathcal{B}$ if there is a bijection from $\mathcal{A}$ to $\mathcal{B}$ that preserves size.

It is important to notice that the combinatorial form of a generating function (3) is simply the reduced representation of the class where the internal structure of each element is destroyed and elements contributing to the object's size (atoms) are replaced by the variable x. As the authors point out, this is analogous to what chemists do when expressing complex molecules into simple molecular formula. For example, isobutane is expressed as C_4H_{10}, which in our approach would be written as x^{14}.

Definition 6. Let Φ be a construction that associates to any collection of classes $\mathcal{B}^1, \mathcal{B}^2, \ldots, \mathcal{B}^m$ a new class

$$\mathcal{A} = \Phi[\mathcal{B}^1, \mathcal{B}^2, \ldots, \mathcal{B}^m]$$

Then the construction is **admissible** if and only if the counting sequence $\{A_n\}$ of $\mathcal{A}$ only depends on the counting sequences $\{B_n^1\}, \{B_n^2\}, \ldots, \{B_n^m\}$ of $\mathcal{B}^1, \mathcal{B}^2, \ldots, \mathcal{B}^m$ and not the internal structure of one or more of the underlying classes $\mathcal{B}^j$.

In this case, there is a well-defined operator Ψ such that

$$A(x) = \Psi[B^1(x), B^2(x), \ldots, B^m(x)]$$

As an example, consider the cartesian product of two classes $\mathcal{B}$ and $\mathcal{C}$. That is, let

$$\mathcal{A} = \mathcal{B} \times \mathcal{C} = \{(b, c) : b \in \mathcal{B}, c \in \mathcal{C}\} \tag{5}$$

where the size of $a = (b, c)$ is defined by

$$|a|_{\mathcal{A}} = |b|_{\mathcal{B}} + |c|_{\mathcal{C}} \tag{6}$$

Notice that the counting sequence of $\mathcal{A}$ is related by the convolution product

$$A_n = \sum_{k=0}^{n} B_k C_{n-k} \tag{7}$$

which means that the cartesian product is an admissible construction. Notice that (7) implies

$$A(x) = B(x)C(x) \tag{8}$$

Specifically, the cartesian product of two classes is admissible and the resulting ordinary generating function is the product of two generating functions.

What about unions? Suppose that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are classes satisfying

$$\mathcal{A} = \mathcal{B} \cup \mathcal{C}, \quad \mathcal{B} \cap \mathcal{C} = \emptyset \tag{9}$$

where

$$|a|_{\mathcal{A}} = \begin{cases} |a|_{\mathcal{B}}, & \text{if } a \in \mathcal{B} \\ |a|_{\mathcal{C}}, & \text{if } a \in \mathcal{C}. \end{cases} \tag{10}$$

It follows that

$$A_n = B_n + C_n \tag{11}$$

so that

$$A(x) = B(x) + c(x) \tag{12}$$

In other words, the union of disjoint sets is admissible and translates to the sum of generating functions.

Example 7. As an (perhaps silly) example, let $\mathcal{B}$ be the class of permutations defined in Example 3 and $\mathcal{C}$ be the class of connected graphs from Example 4.

The two classes are clearly disjoint so that the generating function for $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ is

$$\begin{aligned} A(x) &= B(x) + C(x) \\ &= (x + 2x^2 + 2x^4 + x^5) + (x^2 + x^3 + 3x^4 + x^5) \\ &= x + 3x^2 + x^3 + 5x^4 + 2x^5 \end{aligned}$$

The cartesian product $\mathcal{W} = \mathcal{B} \times \mathcal{C}$ yields

$$\begin{aligned} W(x) &= B(x)C(x) \\ &= (x + 2x^2 + 2x^4 + x^5)(x^2 + x^3 + 3x^4 + x^5) \\ &= x^3 + 3x^4 + 5x^5 + 9x^6 + 5x^7 + 7x^8 + 5x^9 + x^{10} \end{aligned}$$

So, for example, the three objects of size 4 are

(2 1 3 4 5,), (2 1 3 5 4,), (1 2 4 5 3,)

Admissible Constructions

We will define two basic classes, both consisting of a single object. The neutral class $\mathcal{E}$ that consists of a single object of size 0. This is usually denoted $\mathcal{E} = \{\Box\}$ or $\{\epsilon\}$. The reason this is called the neutral class is because

$$\mathcal{A} \cong \mathcal{E} \times \mathcal{A} \cong \mathcal{A} \times \mathcal{E}$$

We also have the atomic class $\mathcal{Z}$ that consists of a single object of size 1. This is usually denoted by $\mathcal{Z} = \{\bullet\}$ or $\mathcal{Z} = \{\circ\}$. We will also choose a single letter, e.g., $\mathcal{Z}_a = \{a\}$, $\mathcal{Z}_b = \{b\}$, etc.

Notice that the generating functions for the neutral and atomic classes are respectively

$$E(x) = 1, \quad Z(x) = x$$

Combinatorial sum (disjoint union): Let $\mathcal{B}$ and $\mathcal{C}$ be combinatorial classes. We wish to define a disjoint sum without any restrictions. There are several ways to accomplish this. For example, we could color the objects in $\mathcal{B}$ blue and the objects in $\mathcal{C}$ could be colored red. We could also create isomorphic copies of $\mathcal{B}$ and $\mathcal{C}$ in such a way that would guarantee their disjointness. We will take this latter approach. So let $\Box$ and $\Diamond$ be neutral objects (of size 0). Then the disjoint union of $\mathcal{B}$ and $\mathcal{C}$ is defined by

$$\mathcal{B} + \mathcal{C} := (\{\Box\} \times \mathcal{B}) \cup (\{\Diamond\} \times \mathcal{C})$$

In this way, the combinatorial sum of two classes is always defined. It follows immediately that

$$\mathcal{A} = \mathcal{B} + \mathcal{C} \quad \Longrightarrow \quad A_n = B_n + C_n \quad \text{and} \quad A(z) = B(z) + C(z)$$

On the other hand, the set theoretical union is not admissible since

$$\text{card}(\mathcal{B} \cup \mathcal{C}) = \text{card}(\mathcal{B}) + \text{card}(\mathcal{C}) - \text{card}(\mathcal{B} \cap \mathcal{C})$$

and the knowledge of the internal structure (in this case, the size of their intersection) is required.

We have already discussed the *cartesian product* (see (5)-(8)).

Sequence construction: If $\mathcal{B}$ is a class then the sequence class $\text{SEQ}(\mathcal{B})$ is defined as the infinite sum

$$\mathcal{A} = \text{SEQ}(\mathcal{B}) = \mathcal{E} + \mathcal{B} + (\mathcal{B} \times \mathcal{B}) + (\mathcal{B} \times \mathcal{B} \times \mathcal{B}) + \cdots \tag{13}$$

where $\mathcal{E}$ is a neutral class. In other words,

$$\mathcal{A} = \{(b_1, b_2, \ldots, b_m) : m \geq 0, b_j \in \mathcal{B}\} \tag{14}$$

Notice that the presence of the neutral class allows one to choose the empty sequence.

Now let $a = (b_1, b_2, \ldots, b_m) \in \mathcal{A}$. Then the size of a is

$$|a| = |b_1| + |b_2| + \cdots + |b_m|$$

Now suppose that $\mathcal{B}$ contains a neutral object, say $\square \in \mathcal{B}$. Now let $\square \neq b \in \mathcal{B}$ with $|b| = k > 0$. Then $\mathcal{A}$ contains an infinite number of objects of size k since each of the following sequences has size k:

$$(b),\ (\square, b),\ (\square, \square, b),\ (\square, \square, \square, b),\ \ldots$$

In this case, SEQ($\mathcal{B}$) is not admissible. Because of this, SEQ($\mathcal{B}$) is admissible if and only if $\mathcal{B}$ does not contain any objects of size 0. *Note:* Let $B(x)$ be the ordinary generating function for the class $\mathcal{B}$. Then the admissibility condition for SEQ($\mathcal{B}$) is equivalent to the generating function stipulation that $B(0) = 0$. This is consistent with our earlier restrictions when dealing with compositions of generating functions.

Now let $A(x)$ be the generating function for $\mathcal{A}$. Then together with (9) and (8), (13) implies

$$A(x) = 1 + B(x) + (B(x))^2 + (B(x))^3 + \cdots \tag{15}$$
$$= \frac{1}{1 - B(x)} \tag{16}$$

Example 8. Let $\mathcal{Z} = \{\bullet\}$ and let $\mathcal{B} = \mathcal{Z} + \mathcal{Z} \times \mathcal{Z}$. Then $\mathcal{B} = \{\bullet, (\bullet, \bullet)\}$, but we will represent this as $\mathcal{B} = \{\bullet, \bullet\!\!-\!\!\bullet\}$ for readability. Then SEQ($\mathcal{B}$) contains

$$\square, \bullet, \bullet\ \bullet, \bullet\!\!-\!\!\bullet,\ \bullet\ \bullet\ \bullet,\ \bullet\ \bullet\!\!-\!\!\bullet,\ \bullet\!\!-\!\!\bullet\ \bullet,\ \bullet\ \bullet\ \bullet\ \bullet,\ \ldots$$

It is interesting to note that the counting sequence for SEQ($\mathcal{B}$) appears to be $1, 1, 2, 3, 5, \ldots$

Notice that $B(x) = x + x^2$, so by (16)

$$A(x) = \frac{1}{1 - (x + x^2)} = \sum_{n \geq 0} f_n x^n$$

where f_n are the Fibonacci numbers.

Exercises

1. Identify the five objects of size 9 of the class $\mathcal{W}$ from Example 7.

2. **Binary Words** - Let $\mathcal{B} = \{a, b\}$ where $|a| = |b| = 1$. Find the first 6 terms in the counting sequence A_n of $\mathcal{A} = \text{SEQ}(\mathcal{B})$. See Example 8.

3. We showed in class that $\mathbb{N} = \text{SEQ}\left(\mathcal{Z}_\bullet\right) \setminus \{\square\}$. Find the generating function for $\text{SEQ}(\mathbb{N})$.

4. Let $\mathcal{Z}_\bullet = \{\bullet\}$ and $\mathcal{B}_{(j,k)} = \underbrace{\mathcal{Z}_\bullet \times \cdots \times \mathcal{Z}_\bullet}_{j \text{ factors}} + \underbrace{\mathcal{Z}_\bullet \times \cdots \times \mathcal{Z}_\bullet}_{k \text{ factors}} = \mathcal{Z}_\bullet^j + \mathcal{Z}_\bullet^k$.

   (a) Find the generating function of $\mathcal{B}_{(2,5)}$ and $\text{SEQ}\left(\mathcal{B}_{(2,5)}\right)$.

   (b) Find the generating function of $\mathcal{B}_{(1,k)}$ and $\text{SEQ}\left(\mathcal{B}_{(1,k)}\right)$.

   (c) In Example 8, we showed that the generating function of $\mathcal{A} = \text{SEQ}\left(\mathcal{B}_{(1,2)}\right)$ was $A(x) = (1 - x - x^2)^{-1}$. Find the generating function for $\text{SEQ}(\mathcal{A})$.