Weighted Generating Functions

Before continuing with admissibility, we need to discuss *weighted* generating functions. We motivate with the following example.

Example 1. Let Subset be the set of all ordered pairs $([n], A)$ where $A \subseteq [n]$, $n > 0$. Let n be the size of $([n], A)$. We leave it as an exercise to show that Subset $\cong \mathrm{SEQ}(\{0,1\})$ with $|0| = |1| = 1$. Now let $S(x)$ be the generating function for Subset. Then

$$S(x) = \frac{1}{1-(x+x)} = \sum_{n\geq 0} 2^n x^n$$

So that the counting sequence for Subset is $\{2^n\}_{n\geq 0}$, as we would expect.

Now suppose we also wish to track the size of A. How might that be accomplished?

For example, consider the pair $([5], \{1,4,5\})$. Using the alphabet $\{0,1\}$, observe that we could map $\{1,4,5\}$ to the "word" 10011. Clearly the length of the word is 5 and the length of the nonzero entries is 3. So we could use a second variable to track the length of the nonzero entries. In the present case, we would assign the **weight** $x^5 y^{|\{1,4,5\}|} = x^5 y^3$ to the pair $([5], \{1,4,5\})$. So in general, we would assign the weight $x^n y^{|A|}$ to the element $([n], A)$. This would correspond to assigning the weights x and xy to the letters 0 and 1 respectively.

It turns out that

$$S_{\mathrm{wt}}(x) = \frac{1}{1-(x+xy)} = \sum_{n\geq k\geq 0} \binom{n}{k} y^k x^n \tag{1}$$

since $\binom{n}{k}$ counts the k-subsets of $[n]$. That is, for $0 \leq k \leq n$, there are $\binom{n}{k}$ ordered pairs of the form $([n], A)$ for $|A| = k$.

We can learn more from this example. Rearranging (1) yields

$$1 = (1-x-xy) \sum_{n\geq k\geq 0} \binom{n}{k} y^k x^n$$

or

$$\sum_{n\geq 0} \binom{n}{k} y^k x^n = x \sum_{n\geq k\geq 0} \binom{n}{k} y^k x^n + xy \sum_{n\geq k\geq 0} \binom{n}{k} y^k x^n$$

It follows that

$$\begin{aligned}\binom{n}{k} = [x^n y^k] \sum_{n\geq k\geq 0} \binom{n}{k} y^k x^n &= [x^n y^k] x \sum_{n\geq k\geq 0} \binom{n}{k} y^k x^n + [x^n y^k] xy \sum_{n\geq k\geq 0} \binom{n}{k} y^k x^n \\ &= [x^{n-1} y^k] \sum_{n\geq k\geq 0} \binom{n}{k} y^k x^n + [x^{n-1} y^{k-1}] \sum_{n\geq k\geq 0} \binom{n}{k} y^k x^n \\ &= \binom{n-1}{k} + \binom{n-1}{k-1}\end{aligned}$$

with $\binom{n}{0} = \binom{n}{n} = 1$. In other words, we have just used generating functions to prove the recursion equation for the binomial coefficients! In the exercises, we ask students to use (1) to prove the Binomial theorem.

The last example suggests the following.

Definition 2. Let $\mathcal{A}_{\text{wt}}$ be a set where each element $a \in \mathcal{A}_{\text{wt}}$ is given a weight $\text{wt}(a)$. This is usually a constant multiple of $x^{|a|}$ or, as we saw in Example 1, a monomial in one or more variables $x_1, x_2, \ldots, x_m$. As before, we require that there only be a finite number of elements of any given weight. Then the **weighted** ordinary generating function of the **weighted** class $\mathcal{A}_{\text{wt}}$ is

$$A_{\text{wt}}(x_1, x_2, \ldots, x_m) = \sum_{a \in \mathcal{A}} \text{wt}(a)$$

See Exercise 4.

Before we continue with the remaining constructions, we remark that if $\mathcal{B}_{\text{wt}}$ and $\mathcal{C}_{\text{wt}}$ are *weighted* classes, then $\mathcal{B}_{\text{wt}} + \mathcal{C}_{\text{wt}}$, $\mathcal{B}_{\text{wt}} \times \mathcal{C}_{\text{wt}}$ and $\text{SEQ}(\mathcal{B}_{\text{wt}})$ are all admissible.

Now for each of the remaining constructions, we let $\mathcal{B}$ be a class with $\Box \notin \mathcal{B}$.

Cycle construction: Cycles are nothing more than sequences taken up to a circular shift. By convention, empty cycles are excluded. We define

$$\text{CYC}(\mathcal{B}) = \text{SEQ}(\mathcal{B}) \setminus \{\Box\}/S$$

where S is the equivalence relation between sequences defined by

$$(b_1, b_2, \ldots, b_m) S (b'_1, b'_2, \ldots, b'_m)$$

if and only if there exists some circular shift σ of $[m]$ such that $b'_j = b_{\sigma(j)}$ for $1 \le j \le m$. For example, let $\mathcal{B} = \{r, g\}$ and consider the 16 sequences from $\text{SEQ}_4(\mathcal{B})$. We list these by equivalence class S:

$$\begin{array}{c} rrrr \\ rrrg\ rrgr\ \ rgrr\ grrr \\ rrgg\ rggr\ \ ggrr\ grrg \\ rgrg\ \ grgr \\ gggr\ ggrg\ \ grgg\ rggg \\ gggg \end{array}$$

In particular, there are only 6 elements in $\text{CYC}_4(\mathcal{B})$.

Multiset construction: Like cycles, multisets are sequences taken up to arbitrary permutations. That is,

$$\text{MSET}(\mathcal{B}) = \text{SEQ}(\mathcal{B})/\Pi$$

where Π is the equivalence relation between sequences defined by

$$(b_1, b_2, \ldots, b_m) \Pi (b'_1, b'_2, \ldots, b'_m)$$

if and only if there exists some permutation π of $[m]$ such that $b'_j = b_{\pi(j)}$ for $1 \le j \le m$. Once again, let $\mathcal{B} = \{r, g\}$ and consider the 16 sequences from $\text{SEQ}_4(\mathcal{B})$. We list these by equivalence class Π:

$$\begin{array}{c} rrrr \\ rrrg\ rrgr\ \ rgrr\ grrr \\ rgrg\ rrgg\ rggr\ \ ggrr\ grrg\ grgr \\ gggr\ ggrg\ \ grgg\ rggg \\ gggg \end{array}$$

Notice that there are only 5 elements in $\text{MSET}(\mathcal{B})$ which is no surprise since $\left(\!\binom{2}{4}\!\right) = \binom{2+4-1}{4} = 5$.

Powerset construction: The powerset class $\mathrm{PSET}(\mathcal{B})$ is defined as the class consisting of all finite subsets of $\mathcal{B}$. Equivalently, $\mathrm{PSET}(\mathcal{B}) \subset \mathrm{MSET}(\mathcal{B})$ without repetitions.

Theorem 3. Let $\mathcal{B}$ and $\mathcal{C}$ be classes. Then the constructions defined above are all admissible. We have

$$\text{(2)} \quad \text{Sum:} \quad \mathcal{A} = \mathcal{B} + \mathcal{C} \quad \Longrightarrow \quad A(x) = B(x) + C(x)$$

$$\text{(3)} \quad \text{Product:} \quad \mathcal{A} = \mathcal{B} \times \mathcal{C} \quad \Longrightarrow \quad A(x) = B(x)C(x)$$

$$\text{(4)} \quad \text{Sequence:} \quad \mathcal{A} = \mathrm{SEQ}(\mathcal{B}) \quad \Longrightarrow \quad A(x) = \frac{1}{1 - B(x)}$$

$$\text{(5)} \quad \text{Powerset:} \quad \mathcal{A} = \mathrm{PSET}(\mathcal{B}) \quad \Longrightarrow \quad A(x) = \prod_{n \geq 1} (1 + x^n)^{B_n} = \exp \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} B(x^k)$$

$$\text{(6)} \quad \text{Multiset:} \quad \mathcal{A} = \mathrm{MSET}(\mathcal{B}) \quad \Longrightarrow \quad A(x) = \prod_{n \geq 1} (1 - x^n)^{-B_n} = \exp \sum_{k \geq 1} \frac{1}{k} B(x^k)$$

$$\text{(7)} \quad \text{Cycle:} \quad \mathcal{A} = \mathrm{CYC}(\mathcal{B}) \quad \Longrightarrow \quad A(x) = -\sum_{k \geq 1} \frac{\phi(k)}{k} \log(1 - B(x^k))$$

where ϕ is the Euler totient function. For the last 4 constructions, we assume that $B(0) = 0$.

Proof: For the combinatorial sum we have

$$A(x) = \sum_{a \in \mathcal{A}} x^{|a|} = \sum_{a \in \mathcal{B}} x^{|a|} + \sum_{a \in \mathcal{C}} x^{|a|} = B(x) + C(x)$$

We have already established the result for cartesian products.

Observe that $B(0) = 0 \implies B(x) = x \sum_{n \geq 1} B_n x^{n-1}$. Now let $f_n(x) = (B(x))^n = x^n \left(\sum_{n \geq 1} B_n x^{n-1} \right)^n$ and notice that $\deg(f_n(x)) \to \infty$ as $n \to \infty$. It follows by Theorem 6 from the Formal Power Series handout and the above rules that

$$A(x) = \sum_{n \geq 0} (B(x))^n$$

is a well-defined power series. Now

$$\begin{aligned} (1 - B(x))A(x) &= (1 - B(x)) \sum_{n \geq 0} (B(x))^n \\ &= \sum_{n \geq 0} (B(x))^n - \sum_{n \geq 0} (B(x))^{n+1} \\ &= 1 + \sum_{n \geq 1} (B(x))^n - \sum_{n \geq 1} (B(x))^n \\ &= 1 \end{aligned}$$

In other words

$$\sum_{n \geq 0} (B(x))^n = \frac{1}{1 - B(x)}$$

which establishes (4).

For the powerset construction, we first suppose that $\mathcal{B}$ is finite with $B_0 = 0$. Then

$$\mathcal{A} = \mathrm{PSET}(\mathcal{B}) \cong \prod_{b \in \mathcal{B}} (\{\square\} + \{b\}) \tag{8}$$

For example, if $\mathcal{B} = \{a, b, c, d\}$ then (8) becomes

$$\mathrm{PSET}(\{a,b,c,d\}) \cong (\{\square\} + \{a\})\,(\{\square\} + \{b\})\,(\{\square\} + \{c\})\,(\{\square\} + \{d\}) \tag{9}$$

Now the subset $\{a, c\}$ would correspond to choosing a from the first factor, c from the third factor, and choosing no elements from the second and fourth factors of the right-hand side of the last expression. The other subsets can be recovered in a similar manner.

It now follows by the sum and product rules above, that

$$A(x) = \prod_{b \in \mathcal{B}} (1 + x^{|b|}) = \prod_{n \geq 1} (1 + x^n)^{B_n}$$

where B_n is the number of elements in $\mathcal{B}$ of size n. Notice that the product is finite since $\mathcal{B}$ is finite.

Now recall from second semester calculus that

$$\log(1 + x) = \sum_{n \geq 1} (-1)^{n+1} \frac{x^n}{n}$$

It follows that

$$\log(1 + x^n) = \sum_{k \geq 1} (-1)^{k+1} \frac{x^{kn}}{k}$$

Thus

$$\begin{aligned}
A(x) &= \prod_{n \geq 1} (1 + x^n)^{B_n} = \exp \log \prod_{n \geq 1} (1 + x^n)^{B_n} \\
&= \exp \sum_{n \geq 1} B_n \log(1 + x^n) \\
&= \exp \sum_{n \geq 1} B_n \sum_{k \geq 1} (-1)^{k+1} \frac{x^{kn}}{k} \\
&= \exp \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \sum_{n \geq 1} B_n \ (x^k)^n \\
&= \exp \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} B(x^k)
\end{aligned}$$

which proves (5) whenever $\mathcal{B}$ is finite. We postpone the discussion of the infinite case.

For the powerset construction we once again suppose that $\mathcal{B}$ is finite and that $B_0 = 0$. Now observe that we can sort any multiset. For example, $S = \{a, b, d, b, c, a, d, b, a\} = \{a, a, a, b, b, b, c, d, d\}$. It now follows immediately that if $\mathcal{A} = \mathrm{MSET}(\mathcal{B})$ then

$$\mathcal{A} = \mathrm{MSET}(\mathcal{B}) = \prod_{b \in \mathcal{B}} \mathrm{SEQ}(\{b\}) \tag{10}$$

That is, if $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ then $\mathcal{A}$ is formed be a sequence of zero or more b_1s, followed by a zero or more b_2s, and so on. It now follows by (4) that

$$\begin{aligned} A(x) &= \prod_{b \in \mathcal{B}} (1 - x^{|b|})^{-1} = \prod_{n \geq 1} (1 - x^n)^{-B_n} \\ &= \exp \log \prod_{n \geq 1} (1 - x^n)^{-B_n} = \exp - \sum_{n \geq 1} B_n \log(1 - x^n) \\ &= \exp \sum_{n \geq 1} \sum_{k \geq 1} B_n \frac{x^{kn}}{k} = \exp \sum_{k \geq 1} \frac{1}{k} \sum_{n \geq 1} B_n (x^k)^n \\ &= \exp \sum_{k \geq 1} \frac{1}{k} B(x^k) \end{aligned}$$

and (6) is established whenever $\mathcal{B}$ is finite. The argument for infinite $\mathcal{B}$ is similar to the one for powerset construction and is omitted.

For the proof of the (7), we refer the interested reader to the text. □

It is useful to allow the admissible constructions identified in Theorem 3 to be restricted in some way. So let $k \geq 0$ and let $\mathcal{B}$ be a class. Then we define the following *restricted* constructions:

$$\mathrm{SEQ}_k(\mathcal{B}) = \underbrace{\mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}}_{k \text{ factors}} = \mathcal{B}^k \tag{11}$$

$$\mathrm{SEQ}_{\leq k}(\mathcal{B}) = \sum_{j=1}^{k} \mathcal{B}^j \tag{12}$$

$$\mathrm{SEQ}_{\geq k}(\mathcal{B}) = \sum_{j \geq k} \mathcal{B}^j = \mathcal{B}^k \times \mathrm{SEQ}(\mathcal{B}) \tag{13}$$

In a similar manner, we will use the symbols $\mathrm{SEQ}_{\text{odd}}$ and $\mathrm{SEQ}_{\text{even}}$ to denote sequences with and odd and even number of components, respectively. The other constructions admit similar restrictive notations.

Now because of Theorem 3, we can immediately specify the corresponding ordinary generating functions for such restricted constructions. For example, (13) immediately yields

$$\mathcal{B}^k \times \mathrm{SEQ}(\mathcal{B}) \quad \Longrightarrow \quad (B(x))^k \frac{1}{1 - B(x)} = \frac{(B(x))^k}{1 - B(x)}$$

Exercises

1. Prove that Subset $\cong$ SEQ($\{0, 1\}$) with $|0| = |1| = 1$ (see Example 1).
   Hint: See the proof of the powerset construction in Theorem 3.

2. Example 1 was not our first encounter with a *weighted* generating function. Consider the collection of unlabeled connected graphs with n vertices. Assign the weight $x^n y^k$ to a graph g with n vertices and k edges.

   (a) Explain why $n - 1 \leq k \leq \binom{n}{2}$ for each n.

   (b) List the weights for each of the graphs of order $1 \leq n \leq 5$. *Note:* By part (a), there are at least 16.

3. Use (1) to prove the Binomial theorem. That is, prove that $(1 + y)^n = \sum_k \binom{n}{k} y^k$.

4. Convince yourself that Definition 2 makes sense by generating all of the terms in the expansion of the right-hand side of (1) for $0 \leq n \leq 3$.

5. List at least 8 elements in each of the following classes. Also, find the corresponding generating functions.

   (a) $b\,\mathrm{SEQ}(a)$

   (b) $\mathrm{SEQ}(b\,\mathrm{SEQ}(a))$

   (c) $\mathrm{SEQ}(a)\,\mathrm{SEQ}(b\,\mathrm{SEQ}(a))$

6. Let $\mathcal{W}^2 = \mathrm{SEQ}(a)\,\mathrm{SEQ}(b\,\mathrm{SEQ}(a))$.

   (a) Identify $\mathcal{W}^2$. List enough elements to see what is going on and find a more direct description.

   (b) What is $\mathcal{W}^1$? Express $\mathcal{W}^3$ in two different ways.