Iterative versus Recursive Constructions

Let us refer to constructions that use Theorem 3 (of the previous lecture) as *iterative*. We will investigate *recursive* constructions below.

Example 1. A few iterative constructions.

a. The natural numbers: $\mathcal{I} = \text{SEQ}(\mathcal{Z}) = \text{SEQ}(\{\bullet\})$ with ogf $I(x) = \dfrac{1}{1-x}$.

b. Binary words: $\mathcal{W} = \text{SEQ}(\mathcal{Z} + \mathcal{Z}) = \text{SEQ}(\{a, b\})$ with ogf $W(x) = \dfrac{1}{1-2x}$.

c. Interval coverings: $\mathcal{F} = \text{SEQ}(\mathcal{Z} + \mathcal{Z} \times \mathcal{Z}) = \text{SEQ}(\{\bullet, \bullet\!\!-\!\!\bullet\})$ with ogf $F(x) = \dfrac{1}{1-x-x^2}$.

d. Compositions: $\mathfrak{C} = \text{SEQ}(\mathcal{I})$ with ogf $C(x) = \dfrac{1}{1-I(x)} = \dfrac{1-x}{1-2x}$

e. Integer partitions: $\mathcal{P} = \text{MSET}(\mathcal{I})$ with ogf $P(x) = \exp \sum_{n \geq 1} \dfrac{I(x^n)}{n}$

Recall that we introduced compositions and integer partitions a few weeks ago. We restate the definitions here for convenience. Let n and k be integers. Then a *composition* of n into k is a sequence $(x_1, x_2, \ldots, x_k)$ of integers such that

$$n = x_1 + x_2 + \cdots + x_k, \qquad x_j \geq 1$$

and a *partition* of n is a sequence $(x_1, x_2, \ldots, x_k)$ of integers such that

$$n = x_1 + x_2 + \cdots + x_k, \qquad x_1 \geq x_2 \geq \cdots \geq x_k \geq 1$$

In the both cases, the x_j's are called summands. We often refer to the summands for partitions as a *weakly decreasing* sequence.

We can visualize each of these using the usual unary representation of the natural numbers

$$\mathcal{I} = \text{SEQ}_{k \geq 1}(\mathcal{Z}) = \{\bullet, \bullet\bullet, \bullet\ \bullet\ \bullet, \ldots\}$$

Then compositions form a unary "ragged landscape" and partitions can be represented by the unary staircase (called a Ferrers diagram) as shown in Figure 1.

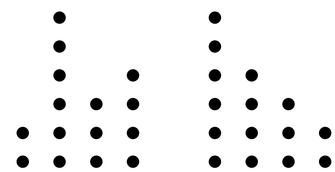


Figure 1: Graphical representation of the composition $2+6+3+4=15$ on the left and the partition $6+4+3+2=15$ on the right.

Now if C_n represents the counting sequence for compositions, then

$$\begin{aligned} C_n &= [x^n]C(x) = [x^n]\,\frac{1-x}{1-2x} \\ &= 2^n - 2^{n-1} = 2^{n-1} \end{aligned}$$

Integer partitions have fascinated mathematicians for centuries. This is no doubt due to the fact that there is no simple formula for the counting sequence P_n of $\mathcal{P} = \text{MSET}(\mathcal{I})$. Now by Theorem 3, we have

$$P(x) = \exp \sum_{n\geq 1} \frac{1}{n} I(x^n) = \exp \sum_{n\geq 1} \frac{x^n/n}{1-x^n} \tag{1}$$

$$= \prod_{n\geq 1} \frac{1}{(1-x^n)^{I_n}} = \prod_{n\geq 1} \frac{1}{1-x^n} \tag{2}$$

Now the product representation yields

$$\begin{aligned} P(x) &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots \\ &= 1+x+2x^2+3x^3+5x^4+7x^5+\cdots \end{aligned}$$

Even though no explicit formula exists for the counting sequence P_n, we have the following famous 1917 result due to Hardy and Ramanujan.

Theorem 2. (Hardy/Ramanujan) Let $P_n = [x^n]\prod_{k\geq 1}(1-x^k)^{-1}$. Then

$$P_n \sim \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{2n/3}}$$

The symbol $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$.

Theorem 3 immediately yields the following results for some *restricted* classes.

Proposition 3. Let $\mathcal{T} \subset \mathcal{I}$ be a subset of the positive integers. Then the ordinary generating functions for the classes $\mathcal{C}^{\mathcal{T}} = \text{SEQ}(\text{SEQ}_{\mathcal{T}}(\mathbb{Z}))$ and $\mathcal{P}^{\mathcal{T}} = \text{MSET}(\text{SEQ}_{\mathcal{T}}(\mathbb{Z}))$ are

$$C^{\mathcal{T}}(x) = \frac{1}{1-T(x)} \quad \text{and} \quad P^{\mathcal{T}} = \prod_{n\in\mathcal{T}} \frac{1}{1-x^n}$$

where $T(x) = \sum_{n\in\mathcal{T}} x^n$ is the ordinary generating function for the class $\mathcal{T}$.

For example, let $\mathcal{T} = \{1, 2\}$. Then $\mathcal{C}^{\mathcal{T}}$ is the set of all compositions with summands restricted to the set $\{1, 2\}$. We have already seen the counting sequence for this class are the Fibonacci numbers since

$$C^{\mathcal{T}}(x) = \frac{1}{1 - T(x)} = \frac{1}{1 - (x + x^2)}$$

More generally we have the following example.

Example 4.

a. Let $\mathcal{T} = \{1, 2, 3, \ldots, r\}$. Then the generating function for all compositions whose summands lie in $\mathcal{T}$ is

$$C^{\mathcal{T}}(x) = \frac{1}{1 - \sum_{k=1}^{r} x^k} = \frac{1}{1 - x\dfrac{1 - x^r}{1 - x}} = \frac{1 - x}{1 - 2x + x^{r+1}} \tag{3}$$

b. Let $\mathcal{T} = \{5, 10, 25\}$. Then there are 29 ways to make change for a dollar using an unlimited supply of nickels, dimes and quarters since

$$[x^{100}]P^{\mathcal{T}}(x) = [x^{100}] \prod_{n \in \mathcal{T}} \frac{1}{1 - x^n} = [x^{100}] \left(\frac{1}{1 - x^5} \frac{1}{1 - x^{10}} \frac{1}{1 - x^{25}} \right) = 29$$

Recursive Constructions

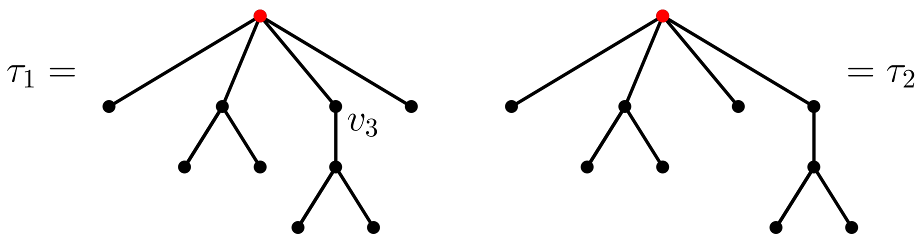
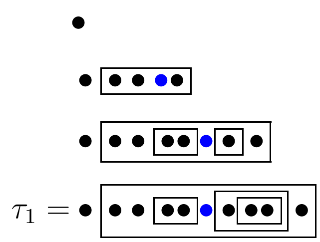


Figure 2: Two distinct plane trees of size 10

Recall that a *tree* is an acyclic, connected graph. A tree is called *rooted* if one of its vertices is identified (the root). A *plane tree* (or ordered tree) is a rooted tree with a specified order assigned to the children of each vertex. Figure 2 shows two distinct plane trees τ_1 and τ_2 of order 10. The root is shown in red and there is an implied order for the four vertices that are adjacent to the root (its children). This implied order is left to right. And the order pattern continues with each descendant.

Notice that we can represent plane trees linearly. We illustrate with τ_1. We build up a linear model recursively. Let $\bullet$ represent the root and let $\bullet\boxed{\bullet\ \bullet\ \bullet\bullet}$ represent the root and each of its children. Continuing, we have



Vertex v_3 is shown in blue throughout. You should find the linear representation of τ_2 and convince yourself that $\tau_1 \neq \tau_2$. (See Exercise 4a).

Now let $\mathcal{G}$ be the class of all plane trees and let $\mathcal{Z} = \{\bullet\}$. Then the above example suggests we can define $\mathcal{G}$ by the recursive equation

$$\mathcal{G} = \mathcal{Z} \times \text{SEQ}(\mathcal{G}) \tag{4}$$

It follows that

$$G(x) = \frac{x}{1 - G(x)}$$

This yields an equation that can solved by radicals.

$$G(x) - (G(x))^2 = x \tag{5}$$

It is now routine to discover that

$$G(x) = \frac{1-\sqrt{1-4x}}{2} = x\,\frac{1-\sqrt{1-4x}}{2x} \tag{6}$$

$$= xC(x) \tag{7}$$

$$= x(1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + 1430x^8 + \cdots) \tag{8}$$

Here $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the ordinary generating function of the Catalan numbers.

Notice that $[x^3]G(x) = 2$ which indicates that there are exactly two plane trees of order 3. See Figure 3.

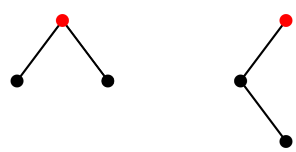


Figure 3: The only plane trees of order 3

On the other hand, there is only one tree of order 3.

Exercises

1. In Figure 1 there appears to be a clear relationship between compositions and partitions of a certain form. Identify this relationship.

2. Let $P(x)$ be the ordinary generating function defined in (2). Use logarithmic differentiation to show that $x\frac{P'(x)}{P(x)} = \sum_{n\geq 1} \frac{nx^n}{1-x^n}$. Use this to show that

$$nP_n = \sum_{k=1}^{n} \sigma(k) P_{n-k} \tag{9}$$

where $\sigma(n)$ is the sum of the divisors of n. For example, $\sigma(6) = 1 + 2 + 3 + 6 = 12$.

Solution:

Let $\mathcal{S}(x) = \sum_n \sigma(n)\, x^n$. Then, by the Wilf rules, the right-hand side of (9) is the counting sequence of the product of the generating functions $S(x)$ and $P(x)$. Now from (2) we have

$$P(x) = \prod_{n \geq 1} \frac{1}{1 - x^n} \tag{10}$$

Logarithmic differentiation yields

$$\frac{P'(x)}{P(x)} = \sum_{n \geq 1} \frac{n x^{n-1}}{1 - x^n} \tag{11}$$

Multiplying by x and rearranging we obtain

$$xP'(x) = P(x) \sum_{n \geq 1} \frac{n x^n}{1 - x^n} \tag{12}$$

So by the Wilf rules, we have

$$\sum_n n P_n\, x^n = xP'(x) = P(x) \sum_{n \geq 1} \frac{n x^n}{1 - x^n} \tag{13}$$

So by (13), it suffices to show that $\mathcal{S}(x) = \sum_{n \geq 1} \frac{n x^n}{1 - x^n}$.

Notice that $[x^n]\mathcal{S}(x) = \sigma(n) = \sum_{m|n} m$. Now suppose that $m \mid n$, say $n = km$ for some $k \geq 1$. Then

$$\begin{aligned} [x^n]\frac{m x^m}{1 - x^m} &= [x^n]\frac{m x^m}{1 - x^m} \\ &= [x^{m(k-1)}]m(1 + x^m + x^{2m} + \cdots) \\ &= m \end{aligned}$$

It is easy to see that if $m \nmid n$, then $[x^n]\frac{m x^m}{1 - x^m} = 0$. The result now follows.

3. Show that

$$[x^n] \sum_{k \geq 0} \left(\frac{x(1 - x^r)}{1 - x} \right)^k = \sum_{j,k} (-1)^k \binom{j}{k} \binom{n - rk - 1}{j - 1}$$

Hint: See (3).

4. Plane trees.

(a) Find a linear representation of the plane tree τ_2 shown in Figure 2.

(b) Sketch the plane tree whose linear representation is given by $\bullet \boxed{\bullet \boxed{\bullet\bullet \boxed{\bullet\bullet}} \bullet \boxed{\bullet\bullet} \bullet \boxed{\bullet \boxed{\bullet\bullet}} \bullet}$.

(c) The nested boxes are not really necessary for the linear representation of a plane tree. Find another way.
Hint: Your answer should help explain the result in (8).

5. Verify the calculations that yield (6). The quadratic equation should have two solutions. What happened to the conjugate solution? Do you recognize $C(x)$ in (8)?

6. From the previous exercise, show that $C(x) = \sum_n \frac{1}{n+1}\binom{2n}{n}x^n$.

7. Consider the recursively defined class $\mathcal{A} = \mathcal{Z}^2 \times \text{SEQ}(\mathcal{A})$ and answer the questions below.

   (a) Find the closed form of ordinary generating function $A(x)$. Express the counting numbers A_n in terms of C_n from the previous exercise.

   Solution:

   Notice that the recursion implies that $A(x)$ must satisfy the functional equation

$$A(x) = \frac{x^2}{1 - A(x)}$$

   The above equation has two solutions

$$A(x) = \frac{1 \pm \sqrt{1 - 4x^2}}{2}$$

   However we can rule out one of these since $A(0) = 0$ (why?). It follows that

$$\begin{aligned} A(x) &= \frac{1 - \sqrt{1 - 4x^2}}{2} \\ &= x^2 \frac{1 - \sqrt{1 - 4x^2}}{2x^2} \\ &= x^2 C(x^2) \end{aligned}$$

   where $C(x)$ is the ordinary generating function for the Catalan numbers as described in (8) and the previous exercise.

(b) What can you say about $\mathcal{A} = \mathcal{Z}^k \times \mathrm{SEQ}(\mathcal{A})$ for a nonnegative integer k?