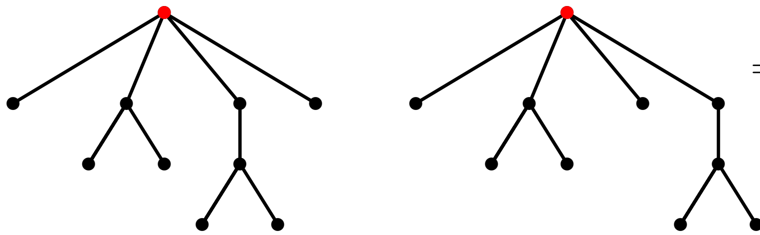
The Lagrange Inversion Formula - Applications

$\tau_1 =$



$= \tau_2$

Figure 1: Two distinct plane trees of size 10

Recall that a *tree* is an acyclic, connected graph. A tree is called *rooted* if one of its vertices is identified (the root). A *plane tree* (or ordered tree) is a rooted tree with a specified order assigned to the children of each vertex. Figure 1 shows two distinct plane trees τ_1 and τ_2 of order 10. The root is shown in red and there is an implied order for the four vertices that are adjacent to the root (its children). This implied order is left to right. And the order pattern continues with each descendant. We should also point out the size of a tree in these definitions is the number of nodes (vertices). We explore another way to define size in the exercises.

Now let $\mathcal{G}$ be the class of all plane trees where we once again measure size of each tree by the number of nodes it possesses. Also, let $\mathcal{Z} = \{\bullet\}$. Earlier in the semester, we argued that $\mathcal{G}$ satisfied the recursion

$$\mathcal{G} = \mathcal{Z} \times \mathrm{SEQ}(\mathcal{G}) \tag{1}$$

Thus

$$G(x) = \frac{x}{1 - G(x)} \tag{2}$$

We went on to show that

$$G(x) = \frac{1 - \sqrt{1-4x}}{2} = x\,\frac{1 - \sqrt{1-4x}}{2x} \tag{3}$$

$$= xC(x) \tag{4}$$

so that

$$[x^n]G(x) = [x^{n-1}]C(x) = c_{n-1} \tag{5}$$

In other words, $g_n = [x^n]G(x) = c_{n-1}$ are the shifted Catalan numbers. Now let $\phi(z) = (1-z)^{-1}$. Then (2) can be rewritten as

$$G(x) = x\phi(G(x))$$

and we may now apply the Lagrange Inversion formula to obtain

$$\begin{aligned}
[x^n]G(x) &= \frac{1}{n}[z^{n-1}](1-z)^{-n} \\
&= \frac{1}{n}[z^{n-1}]\sum_{k\geq 0}\binom{-n}{k}(-1)^k\, x^k \\
&= \frac{1}{n}[z^{n-1}]\sum_{k\geq 0}\binom{n-k-1}{k}\, x^k \\
&= \frac{1}{n}\binom{2n-2}{n-1} = c_{n-1}
\end{aligned}$$

as we saw above. However, one might argue that using LIF is often easier, even when an explicit form of the generating function is known (as in the case above), and the general recursive nature of many types of trees is especially amenable to this technique. We need a few definitions.

Definition 1. A k-ary tree is a plane tree (hence rooted) such the number of children of any node (vertex) is either 0 or k. Except for a few notable exceptions, will use the notation $\mathcal{T}^{\{0,k\}}$ for the class of k-ary trees. *Note:* The classes $\mathcal{T}^{\{0,1\}}$, $\mathcal{T}^{\{0,2\}}$, and $\mathcal{T}^{\{0,3\}}$ are called unary, binary, and ternary trees, respectively.

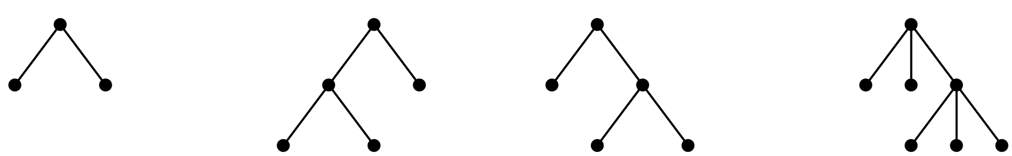


Figure 2: Three binary trees and one ternary tree

We display a few examples in Figure 2. Notice that we continue with convention of displaying the root at the top of the tree. Also, the size of a k-ary tree is the number of nodes. So the respective sizes of the 4 trees shown in Figure 2 are 3, 5, 5, and 7.

We can extend the above definition to include the following.

Definition 2. Let Ω be a finite subset of the natural numbers that contains 0. Then $\mathcal{T}^{\Omega}$ will be the class of plane trees such that the number of children at each node lies in Ω.

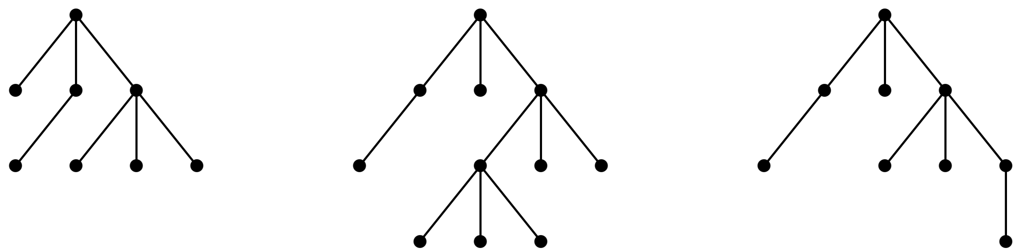


Figure 3: Three trees from the class $\mathcal{T}^{\{0,1,3\}}$

We will refer to such trees as Ω-restricted. Following Flajolet and Sedgewick, we define the *characteristic* function that encapsulates Ω by

$$\phi(z) = \sum_{k \in \Omega} z^k \tag{6}$$

For example, if $\Omega = \{0, 3\}$ then $\phi(z) = 1 + z^3$ is the characteristic function for ternary trees, and the characteristic function of $\Omega = \mathbb{N}$ is $\phi(z) = (1 - z)^{-1}$, i.e., the characteristic function of unrestricted plane trees. We have the following proposition.

Proposition 3. The ordinary generating function $T^{\Omega}(x)$ of the class $\mathcal{T}^{\Omega}$ of Ω-restricted trees satisfies the following recursion

$$T^{\Omega}(x) = x\phi(T^{\Omega}(x)) \tag{7}$$

where ϕ is the characteristic of the set Ω as defined in (6). We also have

$$[x^n]T^{\Omega}(x) = \frac{1}{n}[z^{n-1}]\phi(z)^n \tag{8}$$

Proof: Notice that by the Lagrange Inversion formula, (8) follows immediately from (7), so it is enough to verify (7). If $\mathcal{A}$ is an Ω-restricted sequence, say $\mathcal{A} = \mathrm{SEQ}_{\Omega}(\mathcal{B})$ for some class $\mathcal{B}$, then

$$A(x) = \phi(B(x))$$

Thus

$$\mathcal{T}^{\Omega} = \mathcal{Z} \times \mathrm{SEQ}_{\Omega}(\mathcal{T}^{\Omega}) \quad \Longrightarrow \quad T^{\Omega}(x) = x\phi(T^{\Omega}(x))$$

$\square$

Example 4. Find the counting sequence for the Ω-restricted class $\mathcal{T}^\Omega$ if $\Omega = \{0,1,3\}$. Let $T(x) = T^\Omega(x)$.

Now the characteristic function is $\phi(z) = 1 + z + z^3$, so by Proposition 3 (or the Lagrange Inversion formula),

$$\begin{aligned}
[x^n]T(x) &= \frac{1}{n}[z^{n-1}](1+z+z^3)^n \\
&= \frac{1}{n}[z^{n-1}]\sum_{k=0}^{n}\binom{n}{k} z^{n-k}(1+z^2)^{n-k} \\
&= \frac{1}{n}[z^{n-1}]\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j} z^{n-k+2j} \\
&= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{\frac{k-1}{2}}
\end{aligned}$$

The first few terms in this sequence are $0, 1, 1, 1, 2, 5, 11, 24, 57, 141, 349, 871, \ldots$

Example 5. In 1870, the German mathematician Ernst Schröder asked the following question. In how many ways can n identical variables be "bracketed"? We give a recursive definition: x is a bracketing. And for $k \geq 2$, if $\delta_1, \delta_2, \ldots, \delta_k$ are bracketed expressions, then so is $(\delta_1 \cdot \delta_2 \cdots \delta_k)$. For example, $x, (xx),$ and $(x(xx))$ are bracketed expressions and $(xxx), (x(xx)), ((xx)x)$ are the three bracketings of size 3. If $\mathcal{S}$ is the class of all bracketings, then

$$\mathcal{S} = \mathcal{Z} + \mathrm{SEQ}_{\geq 2}(\mathcal{S}) \quad \Longrightarrow \quad S(x) = x + \frac{S(x)^2}{1 - S(x)} \tag{9}$$

Although the Lagrange Inversion formula does not directly apply to right-hand side of (9), one can solve the equation to conclude

$$S(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4} \tag{10}$$

The counting sequence of $S(x)$ begins with $0, 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, \ldots$

We note that right-hand side of (9) can be rearranged so that the Lagrange Inversion formula applies. We explore this in the exercises.

Exercises

1. Sketch the 11 trees of size 6 from Example 4.

2. Sketch all trees of size 5 for the Ω-restricted trees $\mathcal{T}^{\Omega}$ listed below. Also, find the characteristic function $\phi(z)$, and use Proposition 3 (or the Lagrange Inversion formula) to find a closed form of the counting sequence of each class.

   (a) $\mathcal{T}^{\{0,2\}}$

   (b) $\mathcal{T}^{\{0,1,2,3\}}$

3. In this problem, we investigate the Schröder bracketing problem from Example 5.

   (a) List the 11 bracketings of size 4.

   (b) Use (10) to show that

   $$[x^n]S(x) = \frac{\delta_0(n) + \delta_1(n) + \sum_{k\geq 0} \binom{1/2}{k}\binom{k}{2k-n}(-6)^{2k-n}}{4}$$

   (c) Rearrange the defining equation for $S(x)$ given on the right-hand side of (9) so that the Lagrange Inversion formula can be applied. What is $\phi(z)$?

   (d) Now use the Lagrange Inversion formula to show that

   $$[x^n]S(x) = \frac{1}{n}\sum_{k\geq 0}\binom{2n-k-2}{n-1}\binom{n-2}{k}$$

4. Now let $\mathcal{F}_k$ to the class of *k-ordered* forests defined by $\mathcal{F}_k = \mathrm{SEQ}_k(\mathcal{T})$ where $\mathcal{T}$ is a plane tree and, once again, we measure the size of a forrest by the number of vertices. For example, in $\mathcal{F}_2$, there are zero forests of size 1, one forest of size 2, and two forests of size 3. For the last case, we have $(\bullet, \bullet\!\!-\!\!\bullet)$ and $(\bullet\!\!-\!\!\bullet, \bullet)$. *Note:* There is only one plane tree of size 2, so it is shown here as a barbell: $\bullet\!\!-\!\!\bullet$.

(a) List all of the 2-ordered forests of size 4. There should be five of them.

(b) Let $F_n^k = [x^n]F_k(x)$. According the Lagrange Inversion formula, $F_n^k = [x^n]T(x)^k = \frac{k}{n}[z^{n-k}]\phi(z)^n$. Find a closed formula for F_n^k.

(c) Notice that

$$F_n^k = [x^n]\left(\frac{1-\sqrt{1-4z}}{2}\right)^k \tag{11}$$

Use a CAS (such as MatLab or Wolfram Alpha) to verify (11) for $k \in \{2,3\}$.

5. Let $\mathcal{F}_k = \mathrm{SEQ}_k(\mathcal{T}^\Omega)$. Repeat the part (b) of the previous exercise for each of the following Ω-restricted trees.

(a) $\Omega = \{0,2\}$

(b) $\Omega = \{0,1,2\}$