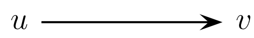


Figure 1: A directed edge

Let G be a simple graph with no loops and let $|V(G)| = d < \infty$. Let $uv \in E(G)$ be an edge. We will use the notation $u \to v$ to indicate the directed edge from u to v. See Figure 1. An orientation ϑ of G is simply the collection of all edges in $E = E(G)$ with each edge given an (arbitrary) assigned direction. Notice that $|\vartheta| = |E|$. An orientation will be called *acyclic* if it contains no directed cycles.

Also, let ϑ be an acyclic orientation on G and let c be an n-coloring of G. We say that c is **compatible** with ϑ if for every directed edge $u \to v$ in ϑ, we have $c(v) \geq c(u)$. We say that the pair is **strictly compatible** if $c(v) > c(u)$.

Theorem 1 (Stanley). Let G be a simple graph with no loops and let $|V(G)| = d < \infty$. Also, let $\chi_G(x)$ be its chromatic polynomial. Then $(-1)^d \chi_G(-n)$ equals the number of compatible pairs (ϑ, c) where ϑ is an acyclic orientation and c is an n-coloring. In particular, $(-1)^d \chi(-1)$ counts the number of acyclic orientations of G.

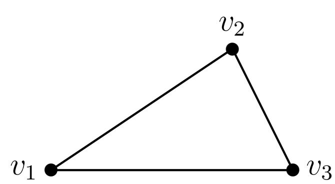


Figure 2: Graph K_3

Before we prove this theorem, it is worthwhile to look at a relevant example.

Example 2. Recall that the chromatic polynomial of the complete graph K_3 shown in Figure 2 is $\chi(x) = x(x-1)(x-2)$. Now according to the above theorem, there are

(a) $(-1)^3 \chi(-1) = 6$ acyclic orientations of K_3.

(b) $(-1)^3 \chi(-2) = 24$ compatible pairs (ϑ, c) where ϑ is an acyclic orientation and $c : V(K_3) \to \{5, 6\}$, i.e., c is a 2-coloring (using the colors 5 and 6). *Note:* The reasons for not using 1 and 2 for colors will become clear below.

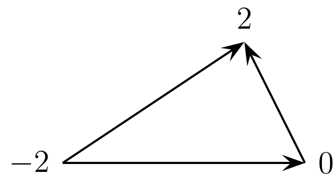


Figure 3: An acyclic orientation ϑ_0 of K_3 using $\rho = \{23\}$

We will sketch 8 of these below. In order to track distinct orientation/coloring pairs, we adopt the following conventions.

(i) The vertex names will be as indicated in Figure 2, but they will not be explicitly marked.

(ii) Each vertex will be encoded with integers indicating the number of arrows directed towards (+) or away (-) from it. For example, v_1 has two arrows directed away from it, hence it's *arrow-encoding* is -2. *Note:* This convention makes it easier to quickly identify different orientations.

(iii) In each sketch, vertices will be either be colored using 5 and 6 or labeled by their arrow-encoding but not both.

We leave it as an exercise to sketch all 6 acyclic orientations. Below we sketch 8 of the 24 compatible orientation/coloring pairs.

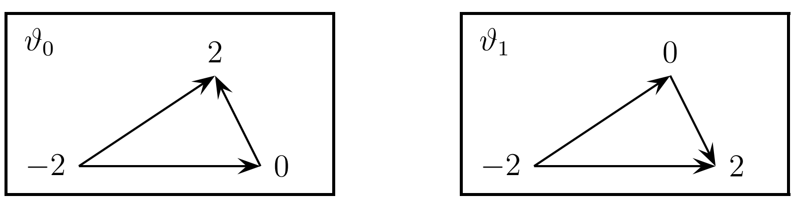
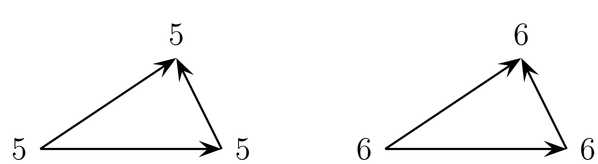


Figure 4: Two acyclic orientations of K_3, ϑ_0 and ϑ_1

Now let's sketch all of the 2-colorings that are compatible to $\vartheta_0 = \{v_1 \to v_2, v_1 \to v_3, v_3 \to v_2\}$. It is easy to see that the two 1-colorings below are compatible with ϑ_0. In fact, 1-colorings are always compatible with acyclic orientations.

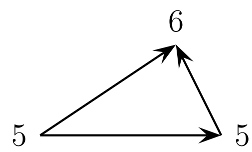


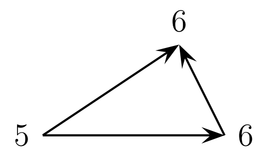
The next two are also compatible. For example, in the sketch on the left below, we have

$$c(v_2) > c(v_1) \text{ and } c(v_2) > c(v_3)$$

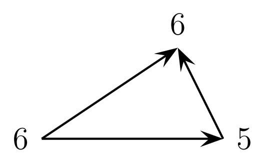
and

$$c(v_3) \geq c(v_1)$$

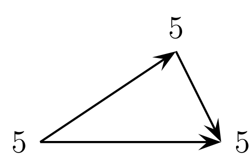


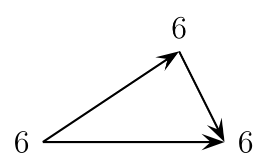


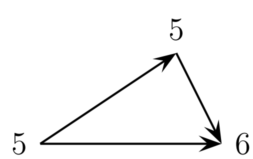
Notice that the coloring below is not compatible with ϑ_0 since $c(v_3) < c(v_1)$.

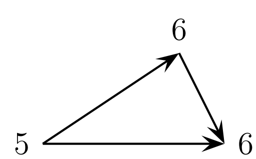


By similar arguments, it should be easy to see that the next 4 colorings are compatible with ϑ_1.









We invite the reader to identify the remaining 16 compatible pairs.

Proof (of Theorem 1): Throughout this proof, an orientation will always mean an *acyclic* orientation. Let $\overline{\chi}_G(n) = (-1)^d \chi_G(-n)$ and let $e = uv \in E(G)$. We have the following contraction/deletion identity for $\overline{\chi}_G$. For $n \in \mathbb{P}$,

$$\begin{aligned}
\overline{\chi}_G(-n) &= (-1)^d \chi_G(n) \\
&= (-1)^d (\chi_{G\setminus e}(n) - \chi_{G/e}(n)) \\
&= (-1)^d \chi_{G\setminus e}(n) + (-1)^{d-1} \chi_{G/e}(n) \\
&= \overline{\chi}_{G\setminus e}(-n) + \overline{\chi}_{G/e}(-n)
\end{aligned}$$

Here the last line follows since $|V(G/e)| = d - 1$. It now follows that

$$\overline{\chi}_G(n) = \overline{\chi}_{G\setminus e}(n) + \overline{\chi}_{G/e}(n)$$

Why?

Now let $\lambda_G(n)$ count the number of compatible pairs (ϑ, c), where ϑ is acyclic orientation and c is an n-coloring. We claim that λ_G satisfies the same contraction/deletion identity as $\overline{\chi}_G$. If the claim is true, then $\lambda_G(n) = \overline{\chi}_G(n)$. To see this, we induct on the size of $E = E(G)$. If $|E| = 0$, then G is the empty graph and

$$\overline{\chi}_G(n) = (-1)^d \chi_G(-n) = (-1)^d(-n)^d = n^d = \lambda_G(n)$$

Now suppose that the result holds for $|E| = k - 1$. Notice that $|E(G \setminus e)| = |E(G/e)| = k - 1$, so that

$$\begin{aligned}\overline{\chi}(n) &= \overline{\chi}_{G\setminus e}(n) + \overline{\chi}_{G/e}(n)\\ &\overset{(*)}{=} \lambda_{G\setminus e}(n) + \lambda_{G/e}(n)\\ &= \lambda_G(n)\end{aligned}$$

Here step (*) holds by induction.

It remains to show that

$$\lambda_G(n) = \lambda_{G\setminus e}(n) + \lambda_{G/e}(n) \tag{1}$$

Let c be an n-coloring of $G \setminus e$. Notice that this also produces an n-coloring of G since $|V(G \setminus e)| = |V(G)|$. Also, let ϑ be an acyclic orientation of $G \setminus e$ compatible with c. Now let ϑ_1 be the orientation of G created by adding the directed segment $u \to v$ to ϑ and let ϑ_2 be the orientation of G created by adding the directed segment $v \to u$ to ϑ. We will show that for each compatible pair (ϑ, c) of $G \setminus e$, exactly one of the pairs (ϑ_1, c) or (ϑ_2, c) is compatible for G, except for $\lambda_{G/e}(n)$ of these pairs, when both are compatible.

i. If $c(u) > c(v)$ then ϑ_2 is compatible with c while ϑ_1 is not. Furthermore, ϑ_2 is acyclic. For if $v \to u \to w_1 \to \cdots \to w_k \to v$ is a directed cycle, then $c(v) < c(u) \le c(w_1) \le \cdots \le c(v)$ which is impossible.

ii. If $c(v) > c(u)$ then ϑ_1 is compatible with c while ϑ_2 is not. Now proceed as in case i.

iii. Finally, if $c(u) = c(v)$, then both ϑ_1 and ϑ_2 are compatible and at least one of them is acyclic. If not, then there exist directed cycles $u \to v \to w_1 \to \cdots \to w_k \to u$ and $v \to u \to w'_1 \to \cdots \to w'_j \to v$. It now follows that

$$v \to w_1 \to \cdots \to w_k \to u \to w'_1 \to \cdots \to w'_j \to v$$

is a directed cycle in ϑ, contrary to our original assumption.

□

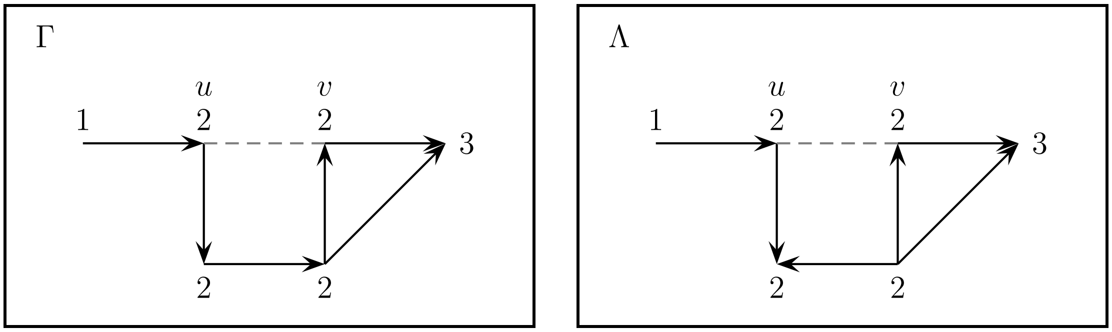


Figure 5: Two compatible pairs (Γ, c) and (Λ, c) for $G \setminus e$

Notice that we show that "at least" one of the orientations in case iii is acyclic. However, Figure 5 makes it clear that there are certain scenarios where adding the directed edge $u \to v$ to an orientation of $G \setminus e$ yields an acyclic orientation of G and adding the directed edge $v \to u$ to an orientation of $G \setminus e$ also produces an acyclic orientation of G. Notice that this occurs precisely when the acyclic orientation on $G \setminus e$ yields an acyclic orientation on G/e. Compare orientations Γ and Λ in Figure 5.

Returning to the notation of item iii above, we suppose that (ϑ, c) is a compatible pair for $G \setminus e$ such that ϑ_1 and ϑ_2 are acyclic orientations of G compatible with c. We define a bijection $\Phi(\vartheta, c) = (\vartheta', c')$ as follows. Let x be the vertex in G/e obtained by identifying u with v (see Fig. 6). Since $E(G \setminus e) = E(G/e)$, we define ϑ' by $w_1 \to w_2$ in ϑ if and only if $w_1 \to w_2$ in ϑ' and we define $c'(w) = c(w)$ for each $w \in V(G/e) \setminus x$ and $c'(x) = c(u) = c(v)$. It is clear that $\Phi(\vartheta, c) = (\vartheta', c')$ is the desired bijection.

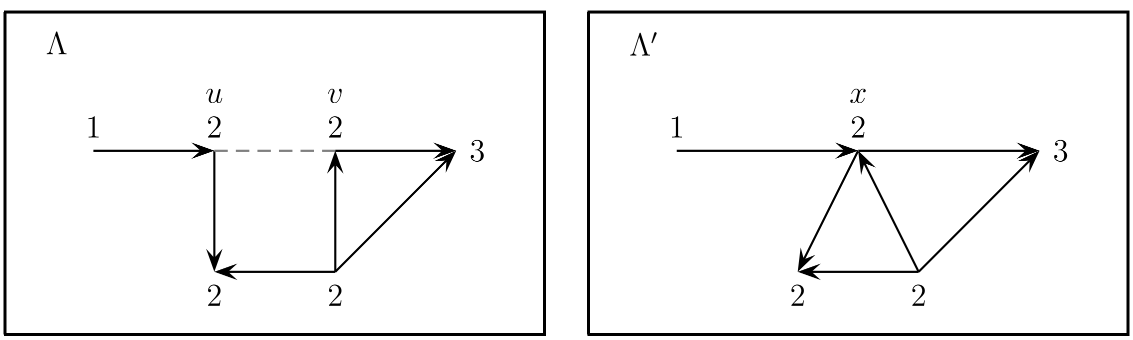


Figure 6: Compatible pair (Λ, c) for $G \setminus e$ and the corresponding image under Φ of the compatible pair (Λ', c') for G/e

Now since the number of compatible pairs in G/e is $\lambda_{G/e}(n)$, the identity in (1) is established.

The above proof follows closely the proof of Theorem 1.2 given in Stanley's paper. Except is noted in the introduction (and in class), our definitions of orientations, compatible pairs, etc. are as described in section 1.1 of Beck and Sanyal's monograph Combinatorial Reciprocity Theorems. In particular, we avoid the awkward method of describing an orientation using the exception set ρ, preferring instead to simply list each of the directed edges. See ϑ_0 in Example 2 above.