1. (4 points) Prove that the number of integer partitions on n into exactly k parts is equal to the number of partitions of n in which the largest part is k.

   Solution:

   Let $P([n])$ be the set of integer partitions. Then the map $T : P([n]) \longrightarrow P([n])$ defined by $T(\lambda) = \lambda^t$ that sends the partition λ to its transpose is clearly a bijection since $T^2(\lambda) = \lambda$. In particular, if $\lambda \vdash n$ has exactly k parts, then the largest part of $\lambda^t = (\lambda_1^t, \lambda_2^t, \ldots, \lambda_m^t)$ is λ_1^t which is equal to k. Since T is a bijection, we are done.

2. (4 points) Let $p(n)$ be the number of integer partitions of n. Prove that for all $n \geq 2$, the number $q(n) = p(n) - p(n-1)$ is equal to the number of integer partitions of n in which the largest parts are equal. *Hint:* According to the text, $q(n)$ is also equal to the number of integer partitions in which the smallest block has size at least two.

   Solution:

   Note: The hint refers to Theorem 5.20 from the text.

   Continuing with the notation above, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ with $\lambda_1 = \lambda_2 = m$, then λ^t has exactly m parts with $\lambda_m^t \geq 2$ (else $\lambda_1 \neq \lambda_2$). In other words, the set of integer partitions with equal largest parts is is isomorphic to the set of integer partitions with smallest block size of at least two. The result now follows from Theorem 5.20.

$n \backslash k$	0	1	2	3	4	5	6	7	r_n
0	1								1
1	0	1							1
2	0	1	2						3
3	0	1	7	6					14
4	0	1	18	46	24				89
5	0	1	41	228	326	120			716
6	0	1	88	930	2672	2556	720		6967
7	0	1	183	3406	17198	31484	22212		79524

Table 1: Silly numbers $\left|{n \atop k}\right|$

3. Let n and k be integers. We define the *silly* coefficients $\left|{n \atop k}\right|$ by the following recursion

$$\left|{n+1 \atop k}\right| = k\left|{n \atop k}\right| + (n+1)\left|{n \atop k-1}\right| \tag{1}$$

together with boundary conditions $\left|{0 \atop 0}\right| = 1$ and $\left|{n \atop k}\right| = 0$ whenever $n < 0$ or $k < 0$ or $k > n$. We list a few values in Table 1.

(a) (4 points) Show that $S_1(x) = \sum_{n\geq 0} \left|{n \atop 1}\right| \frac{x^n}{n!} = e^x - 1$.

Solution:

We could appeal to the fact that $\left|{n \atop 1}\right| = 1$ for $n > 0$ (See Table 1). Instead, let's use the fact that $\left|{0 \atop 0}\right| = 1$, $\left|{n \atop 0}\right| = 0$ for $n > 0$, and the recurrence (1). It follows that $S_0(x) = \sum_n \left|{n \atop k}\right| x^n/n! = 1$ and

$$\begin{aligned} S_1'(x) &= \sum_n \left|{n+1 \atop 1}\right| \frac{x^n}{n!} \\ &= \sum_n 1\left|{n \atop 1}\right| \frac{x^n}{n!} + \sum_n (n+1)\left|{n \atop 0}\right| \frac{x^n}{n!} \\ &= S_1(x) + 1 \end{aligned}$$

Rearranging the last equation and multiplying by the integerating factor, e^{-x} produces

$$D(e^{-x}S_1(x)) = e^{-x}$$

Integrating both sides yeilds

$$e^{-x}S_1(x) = -e^{-x} + C = -e^{-x} + 1 \qquad \text{(since } S_1(0) = 0\text{)}$$

After rearranging we obtain

$$S_1(x) = -1 + e^x$$

as expected.

(b) (8 points) Find the closed form of the exponential generating function $S_2(x) = \sum_{n\geq 0} \left|{n \atop 2}\right| \frac{x^n}{n!}$. *Hint:* We saw an example of how to do this on Wednesday when we showed that $\sum_{n\geq 0} \binom{n}{k} \frac{x^n}{n!} = x^k\, e^x/k!$. *Warning:* The closed form of $S_2(x)$ isn't a particulary nice function.

Solution:

Proceeding much as we did above, we have

$$\begin{aligned}
S_2'(x) &= \sum_n \left|\begin{matrix} n+1 \\ 2 \end{matrix}\right| \frac{x^n}{n!} \\
&= 2\sum_n \left|\begin{matrix} n \\ 2 \end{matrix}\right| \frac{x^n}{n!} + \sum_n (n+1) \left|\begin{matrix} n \\ 1 \end{matrix}\right| \frac{x^n}{n!} \\
&= 2S_2(x) + \sum_n n \left|\begin{matrix} n \\ 1 \end{matrix}\right| \frac{x^n}{n!} + \sum_n \left|\begin{matrix} n \\ 1 \end{matrix}\right| \frac{x^n}{n!} \\
&= 2S_2(x) + xD(S_1(x)) + S_1(x) \qquad \text{(by Rule 3}'\text{)} \\
&= 2S_2(x) + xe^x + e^x - 1
\end{aligned}$$

Thus

$$D(e^{-2x}S_2(x) = xe^{-x} + e^{-x} - e^{-2x}$$

Integrating both sides produces

$$e^{-2x}S_2(x) = \frac{e^{-2x}}{2} - (x+2)e^{-x} + C = \frac{e^{-2x}}{2} - (x+2)e^{-x} + \frac{3}{2}$$

or

$$S_2(x) = \frac{1}{2} - (x+2)e^x + \frac{3e^{2x}}{2}$$

Using any CAS, one can compare the first few terms in the sequence of coefficients of $S_2(x)$ to the relevant column in Table 1 and confirm that they are in agreement.

$$0, 0, 2, 7, 18, 41, 88, 183, 374, 757, 1524, 3059, \ldots$$