1. Recall that if we let Let $\mathcal{Z} = \{\bullet\}$ with $\bullet$ an atom (of size 1). Then $\mathcal{I} = \mathrm{SEQ}(\mathcal{Z}) \setminus \{\square\} = \{\bullet, \bullet\bullet, \bullet\bullet\bullet, \ldots\}$ is combinatorial way to describe the positive integers in unary notation.
Note: It is worth noting that the counting sequence of $\mathcal{I}$ is $\{0, 1, 1, 1, \ldots\}$.

   (a) (6 points) Find the closed form of the ordinary generating function $I(x)$ of $\mathcal{I}$. THIS IS NOT A TRICK QUESTION.

   Solution:

   $I(x) = \dfrac{x}{1-x}$

   (b) (7 points) Let k be a positive integer and let

   $$\mathfrak{C}^{(k)} = \mathrm{SEQ}_k(\mathcal{I}) = \underbrace{\mathcal{I} \times \mathcal{I} \times \cdots \times \mathcal{I}}_{k \text{ factors}}$$

   Find the closed form of the ordinary generating function $C^{(k)}(x)$ of $\mathfrak{C}^{(k)}$.

   Solution:

   By the product rule, $C^{(k)}(x) = (I(x))^k = \dfrac{x^k}{(1-x)^k}$

(c) (7 points) Find $C_n^{(k)} = [x^n]C^{(k)}(x)$.

Solution:

$$\begin{aligned} C_n^{(k)} &= [x^n]\,\frac{x^k}{(1-x)^k} \\ &= [x^n]x\,\frac{x^{k-1}}{(1-x)^k} \\ &= [x^{n-1}]\,\frac{x^{k-1}}{(1-x)^k} \\ &= \binom{n-1}{k-1} \end{aligned} \tag{1}$$

Remark. The result should look familiar. For $n \geq k$, $C_n^{(k)}$ should count the number of ways to assign n votes to k candidates so that each candidate receives at least one vote since $\square \notin \mathcal{I}$ (see Figure 1).

$$\begin{aligned} C_n^{(k)} &= \left(\!\!\binom{k}{n-k}\!\!\right) \\ &= \binom{n-k+k-1}{n-k} \\ &= \binom{n-1}{n-k} \end{aligned}$$

which is (1).

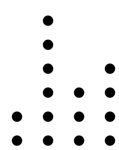

Figure 1: Graphical representation of 15 votes distributed between 4 candidates with each candidate receiving at least one vote.