Hint or Answer to Some Homework Problem in Section 3.2

2. (a) Two of them are the identity matrix and the zero matrix. This may also help you to find other two.

(c) You have $e^2 - e = 0$ and therefore $e(e-1) = 0$. Then you use the definition of integral domain.

5. (a). Take a, b in $S \cap T$. You need to prove both $a - b$ and ab in $S \cap T$. To get these, to prove that $a - b$ and ab are in both S and T.

(b). No. For example, as $S = 2\mathbb{Z}$ and $T = 3\mathbb{Z}$.

6. This is an easy problem. Just use Theorem 3.2 or 3.6.

11. (a) You prove $(ab)(b^{-1}a^{-1}) = 1_R$ and $(b^{-1}a^{-1})(ab) = 1_R$, and then use definition.

(b) No. For counterexample, consider two invertible matrices A and B such that $AB \neq BA$.

12. (a) The fact that $[a]$ is a unit in $\mathbb{Z}_n$ is equivalent to that the equation $[a]x = 1$ has a solutoin in $\mathbb{Z}_n$.

For the "if" part, use Corollary 2.10 with $b = 1$.

For the "only if" part, you note that $[a]x = 1$ has a solutoin in $\mathbb{Z}_n$ means that there is $k \in \mathbb{Z}$ such that $ax = 1 + kn$ or $ax - kn = 1$. Use this, try to prove $(a, n) = 1$.

(b) Use the same arguments as in the proof of Theorem 3.11.

Consider the products $[a][1], [a][2], \ldots, [a][n-1]$. If they are all different, then $[a]$ is a unit. If $[a][i] = [a][j]$ for some $i \neq j$, then $[a]$ is a zero divisor.

Of course you need to organize your proof in a good way.

26. **Answer:** $ab = 1_R = ba$, $aa = b$ and $bb = a$.

To get this, you need to know that a is a unit implies that a^{-1} exists. Hence, $a0_R$, $a1_R$, aa, ab are all different. Then with some work, you can conclude that $ab = 1_R$ and $aa = b$ are the only possibility.