3. Limit Sets and Topological Conjugacy

Let X be a compact metric space with metric d and let $f : X \to X$ be a homeomorphism.

For any subset $\Lambda \subset X$ and $n \in \mathbb{Z}$, denote $f^n(\Lambda) = \{f^n(x) : x \in \Lambda\}$.

Definition 2.1. *A subset $\Lambda \subset X$ is* invariant *for f if $f(\Lambda) = \Lambda$.*

The orbit $O(x)$ of x is the set $\{f^n(x) : n \in \mathbb{Z}\}$. The *forward orbit* $O_+(x)$ is the set $\{f^n(x) : n \in \mathbb{Z}_+\}$, and the *backward orbit* $O_-(x)$ is the set $\{f^n(x) : n \in \mathbb{Z}_-\}$. We say that x is periodic if there is a positive integer τ such that $f^\tau(x) = x$. The least such τ is called the period of x.

Clearly $O(x)$ is an invariant set.

Definition 2.2. *Let $x \in X$.*

The ω-limit set *of x, denoted by $\omega(x)$, is the set of points y such that there is a sequence $n_1 < n_2 < \dots$ with $n_i \to +\infty$ and $f^{n_i}(x) \to y$ as $i \to \infty$.*

The α-limit set *of x, denoted by $\alpha(x)$, is the set of points y such that there is a sequence $n_1 > n_2 > \dots$ with $n_i \to -\infty$ and $f^{n_i}(x) \to y$ as $i \to \infty$.*

Lemma 2.3. *For any $x \in X$, $\omega(x)$ and $\alpha(x)$ are closed invariant subsets.*

Proof. (Invariance) We prove $f(\omega(x)) = \omega(x)$. Let $y \in \omega(x)$. We can take $n_1 < n_2 < \dots$ such that $n_i \to \infty$ and $f^{n_i}(x) \to y$ as $i \to \infty$. Hence, by continuity we have $n_1 \pm 1 < n_2 \pm 1 < \dots$ such that $n_i + 1 \to \infty$ and $f^{n_i \pm 1}(x) = f^{\pm 1}(f^{n_i}(x)) \to f^{\pm 1}(y)$ as $i \to \infty$. So $f^{\pm 1}(y) \in \omega(x)$. Since $y \in \omega(x)$ is arbitrary, we get $f^{\pm 1}(\omega(x)) \subseteq \omega(x)$. That is, $f(\omega(x)) \subseteq \omega(x)$ and $f^{-1}(\omega(x)) \subseteq \omega(x)$, while the latter implies $\omega(x) \subseteq f(\omega(x))$.

(Closeness) Suppose $\{y_k\} \subset \omega(x)$ such that $y_k \to y_0$ as $k \to \infty$. For each $k > 0$, we can choose $n_k > 0$ inductively such that $n_k > \min\{n_{k-1}, k\}$ and $d(f^{n_k}(x), y_k) \le d(y_k, y_0)$. We get a sequence $n_1 < n_2 < \dots$ with $n_k \to \infty$. Since

$$d(f^{n_k}(x), y_0) \le d(f^{n_k}(x), y_k) + d(y_k, y_0) \le 2d(y_k, y_0) \to 0,$$

we have $f^{n_k}(x) \to y_0$ as $k \to \infty$. So $y_0 \in \omega(x)$, and therefore $\omega(x)$ is a closed subset. □

Let $P(f)$ denote the set of periodic points of f.

Define the *negative* and *positive limit sets* of f by

$$L_-(f) = \overline{\bigcup_{x \in X} \alpha(x)}, \qquad L_+(f) = \overline{\bigcup_{x \in X} \omega(x)},$$

where $\overline{S}$ meanse the closure of a subset S. Then define the *limit set* of f by $L(f) := L_-(f) \cup L_+(f)$.

Definition 2.4. *A point $x \in X$ is a* nonwandering point *of f if for any neighborhood U of x and any $N > 0$, there exists $n > N$ such that*

$$f^n(U) \cap U \neq \emptyset.$$

The set of all nonwandering points is called the nonwandering set *and is denoted by $\Omega(f)$ or* NW(f)*.*

It can be proved that a point $x \in X$ is a nonwandering point if for any neighborhood U of x, there exists $n > 1$ such that $f^n(U) \cap U \neq \emptyset$.

By the definition, $x \in X$ is a nonwandering point if for any neighborhood U of x, there exists $y \in U$ such that $f^n(y) \in U$ for some $n > 1$. Hence, any point $y \in \alpha(x)$ or $\omega(x)$ is a nonwandering point and therefore $L(f) \subset \Omega(f)$ by the lemma below.

Lemma 2.5. *A nonwandering set is a closed invariant set.*

Proof. Invariance is clear.

Note that $x \in X$ is a wandering point if there is a neighborhood U of x such that for any $N > 0$, $f^n(U) \cap U = \emptyset$. It means that all points in U are wandering point. So the set of wandering points are open, and hence nonwandering set is closed. □

Definition 2.6. *A subset Σ is called a* minimal set *of f if it is a nonempty closed invariant set that does not contain any closed invariant proper subset.*

Denote by Min(f) the union of all minimal set of f.

Proposition 2.7. *Any compact invariant set contains a minimal set.*

Proof. Let K be a compact invariant set. Let $\mathcal{S}$ be the collection of all nonempty invariant compact subset A of K. Define a partial order "$\prec$" in $\mathcal{S}$ by $A \prec B$ if $A \subset B$. Then by Zorn's lemma, every linearly ordered subsets $\cdots \prec A_i \prec A_{i-1} \prec \dots$ has a least element Σ. In fact, Σ is the intersection of the set $\{A_i\}$, and is nonempty by Cantor intersection theorem. Σ is a minimal set. □

Lemma 2.8. *A compact subset Σ is minimal if and only if for any $x \in \Sigma$, $\overline{O(x)} = \Sigma$.*

Proof. "$\Longrightarrow$": Since $\overline{O(x)}$ is a nonempty colsed invariant set and $\overline{O(x)} \subseteq \Sigma$, minimality of Σ implies $\overline{O(x)} = \Sigma$.

"$\Longleftarrow$": If Σ is not a minimal set, then there is a nonempty closed invariant subset Σ_1 properly contained in Σ, then for any $x \in \Sigma_1$, $\overline{O(x)} \subseteq \Sigma_1 \subsetneqq \Sigma$, a contradiction. □

Definition 2.9. *An* ϵ-chain *is a finite sequence* $x_1, x_2, \ldots, x_n$ *in* X *such that* $d(f(x_i), x_{i+1}) < \epsilon$ *for* $1 \le i \le n-1$.

A point x *is* chain recurrent *if for every* $\epsilon > 0$ *there is an* ϵ*-chain starting and ending at* x.

The chain recurrent set *of* f*, denoted by* $R(f)$ *or* $\mathrm{Rec}(f)$*, is the set of chain recurrent points.*

Lemma 2.10. (1) *A chain recurrent set is a closed invariant set.*
(2) *A nonwandering point is a chain recurrent point, and hence* $R(f) \subseteq \Omega(f)$.

Definition 2.11. *A point* x *is* forward recurrent *if* $x \in \omega(x)$*, and* backward recurrent *if* $x \in \alpha(x)$*. It is* recurrent *if it is both forward and backward recurrent.*

The Birkhoff Center*, denoted by* $C(f)$ *or* $\mathrm{BC}(f)$*, is the closure of the set of recurrent points.*

Lemma 2.12. (1) *The* Birkhoff Center $C(f)$ *is invariant.*
(2) *For a recurrent point* x, $x \in \omega(x) \cap \alpha(x) \subseteq L(f)$*, and hence* $C(f) \subseteq L(f)$.

Summarizing the above relations we have

$$\overline{P(f)} \subseteq \mathrm{Min}(f) \subseteq \mathrm{BC}(f) \subseteq L(f) \subseteq \mathrm{NW}(f) \subseteq \mathrm{Rec}(f).$$

Example 2.13. (1) *Any fixed point or periodic orbit is a minimal set by Lemma 2.8.*
(2) *Let* $f : \mathbb{S}^1 \to= \mathbb{S}^1$ *be a homeomorphism that contains exact one fixed point p as in Figure 1. Then* $\Omega(f) = \{p\}$ *and* $R(f) = \mathbb{S}^1$.

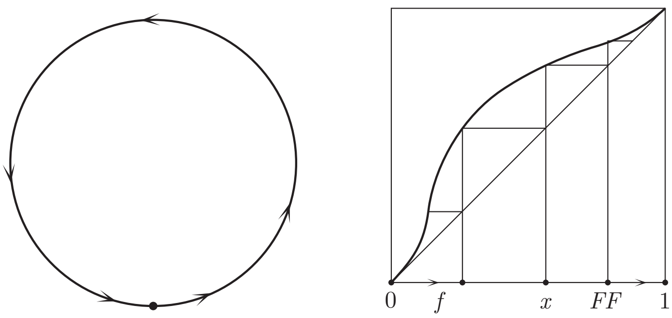

Figure 1. Circle map with one fixed point

(3) *Let* $R_\alpha : \mathbb{S}^1 \to= \mathbb{S}^1$ *be a circle rotation given by* $R_\alpha(z) = e^{i\alpha}z$*, where* $z \in \mathbb{S}^1$.

If α *is a rational number, then for any* z, $\alpha(z) = \omega(z) = O(z)$ *is a periodic orbit. Hence,* $P(R_\alpha) = \mathbb{S}^1$.

If α is an irrational number, then for any z, $O(z)$ is dense on $\mathbb{S}^1$. Hence $P(R_\alpha) = \emptyset$ and $\alpha(z) = \omega(z) = \Sigma = \mathbb{S}^1$ is α- and ω-limit set and the minimal set containing z.

(4) *(The Mathematical Pendulum) The phase portrait of a mathematical pendulum is given in Figure 2, which is a continuous dynamical system. The orbits connect critical points are called the* heteroclinic orbits. *The heteroclinic orbits are not in the limit set, but in the nonwandering set of the flow.*

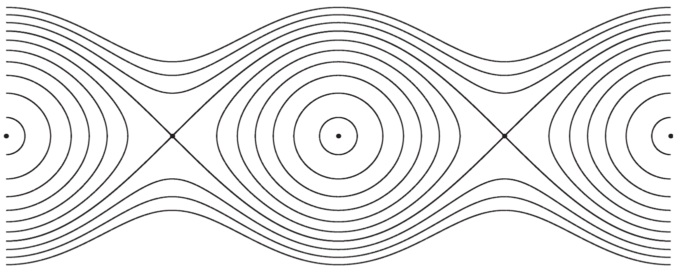

FIGURE 2. Phase portrait of the mathematical pendulum

Definition 2.14. *Suppose $f : X \to X$ and $g : Y \to Y$ are two homeomorphisms of metric spaces X and Y respectively. We say that f is* topologically conjugate *to g if there is a homeomorphism $h : X \to Y$ such that $hf = gh$. Any such h is called a* topological conjugacy.

We say that f is topologically semiconjugate *to g, or g is a* factor *of f, if there is a continuous map $h : X \to Y$ such that $hf = gh$.*

Lemma 2.15. *The topological conjugacy relation is an equivalence relation on any given set of homeomorphisms of metric spaces, that is,*

(1) *$f : X \to X$ is topological conjugate to itself;*
(2) *If $f : X \to X$ is topological conjugate to $g : Y \to Y$, then $g : Y \to Y$ is topological conjugate to $f : X \to X$;*
(3) *If $f : X \to X$ is topological conjugate to $g : Y \to Y$, and $g : Y \to Y$ is topological conjugate to $k : Z \to Z$, then $f : X \to X$ is topological conjugate to $k : Z \to Z$.*

Lemma 2.16. *Suppose $f : X \to X$ is topologically conjugate $g : Y \to Y$ with a topological conjugacy $h : X \to Y$. Then*

(1) *for any $x \in X$, $h(O_f(x)) = O_g(h(x))$;*
(2) *$h(\mathrm{RS}(f)) = \mathrm{RS}(g)$, where $\mathrm{RS} = \overline{P}, \mathrm{Min}, \mathrm{BC}, L, \mathrm{NW}$ or Rec.*

Proof. The proof is based on the facts that $hf^n = g^n h$ for any $n \in \mathbb{Z}$ and continuity of h. □