MATH 848 **Homework 1** **Fall, 2014**

1. Suppose $|\cdot|'$, $|\cdot|''$ are two norms on $\mathbb{R}^n$. Prove that there are constants $C_1 > 0$, $C_2 > 0$ such that for every $x \in \mathbb{R}^n$,

$$C_1|x|' \le |x|'' \le C_2|x|'$$

2. Consider the norm $|\cdot|_p$ and $|\cdot|_\infty$ on $\mathbb{R}^n$ defined by

$$|x|_p = \Big(\sum_{1\le i\le n} |x_i|^p\Big)^{1/p}, \qquad |x|_\infty = \sup_{1\le i\le n} |x_i|,$$

where $p \ge 1$. Prove that $\lim_{p\to\infty} |x|_p = |x|_\infty$.

3. Suppose $T : X \to Y$ is a linear map of a Banach space X into Banach space Y . Let

$$\begin{aligned} A &= \inf\{k : |Tx| \le k|x| : \forall x \in X\} \\ B &= \sup_{x\ne 0}\Big\{\frac{|Tx|}{|x|}\Big\} \\ C &= \sup_{|x|\le 1}\{|Tx|\} \end{aligned}$$

Show that $A = B = C$.

4. Suppose $T : X \to Y$ is a one-to-one continuous onto linear map from the Banach space X to the Banach space Y and there is a constant $k > 0$ such that $|Tx| \ge k$ for all $|x| = 1$. Prove that there is a unique continuous linear map $S : Y \to X$ such that $S(Tx) = x$ for all x.

5. Let $I = [0, 1]$ be the closed unit interval. Show that the closed unit ball in the Banach space $C(I, \mathbb{R}^n)$ is not compact.

6. Show that the Schauder Fixed Point Theorem becomes false if either of the compactness or convexity conditions does not hold.

7. Let F be an arbitrary closed subset of $I = [0, 1]$. Show that there is a strictly increasing continuous function ϕ from I to I such that the set of fixed points of ϕ is precisely F.

8. Let $I \in \mathbb{R}$ be an open interval and $f \in C^1(I, \mathbb{R})$. let $a \in I$ s.t. $f'(a) \neq 0$.

(a) Show that $\exists \varepsilon > 0$ such that $\forall x \in W := (a - \varepsilon, a + \varepsilon)$, $\forall y \in \mathbb{R}$, the function $\phi : W \to \mathbb{R}$ given by

$$\phi_y(x) = x + \frac{y - f(x)}{f'(a)}$$

is a contraction map with contraction constant $\lambda \leq 1/2$.

(b) Show that $\exists \delta > 0$ s.t. $\forall y \in V := (f(a) - \delta, f(a) + \delta)$, $\phi_y(W) \subset W$.

(c) Prove that for any $y \in V$, there is a unique $x \in W$ such that $f(x) = y$ and x depends on y continuously. (This is a part of *inverse function theory.*)

9. Prove the following Contraction Mapping Theorem: Suppose $\mathcal{X}$ is a complete metric space and $T : \mathcal{X} \to \mathcal{X}$ is a map such that T^n is a contraction map for some $n \in \mathbb{N}$. Then T has a unique fixed point $x^* \in \mathcal{X}$.

10. (a) Suppose $f : I \to \mathbb{R}$ is continuous and $f(I) \supset I$, where $I = [0, 1]$. Show that f has a fixed point in I.

(b) A point x of a map f is *periodic* if it is a fixed point of f^n for some $n \in \mathbb{N}$, and such n is called the *period*, where f^n is the nth iteration of f. Prove that if a continuous map $f : I \to I$ has a periodic point of period 3, then f has a periodic point of any period $n \in \mathbb{N}$. (This result is contained in *Sarkovskii's Theorem* and *Li-Yorke Theorem.*)