MATH 848 Exercise Set 2 Fall, 2014

1. Show that the initial value problem $\dot{x} = x/t$, $x(0) = 0$, has infinitely many solutions and the equation for the initial value $x(0) = c$, $c \neq 0$, has no sulution.

2. Suppose that the function $f(t,x)$ is bounded and continuous on $\mathbb{R} \times \mathbb{R}^n$ and has continuous partial derivatives with respect to x. Prove that any solution $x = \phi(t)$ of the equation $\dot{x} = f(t,x)$ is defined for all $t \in (-\infty, \infty)$.

3. Prove the following Comparison Theorem: Suppose $f(t,x)$ and $g(t,x)$ are continuous functions defined on an open set $D \subset \mathbb{R} \times \mathbb{R}$ with $f(t,x) < g(t,x)$ for all $(t,x) \in D$, and $\phi(t)$ and $\psi(t)$ are solutions of $\dot{x} = f(t,x)$ and $\dot{x} = g(t,x)$, respectively, satisfying $x(t_0) = x_0$, where $(t_0, x_0) \in D$. Then

(a) $\phi(t) < \psi(t)$ if $t > t_0$ and both $\phi(t)$ and $\psi(t)$ are defined.

(b) $\phi(t) > \psi(t)$ if $t < t_0$ and both $\phi(t)$ and $\psi(t)$ are defined.

4. Proof Claim 2 in Proof 2 of the Peano's Theorem. That is, on $I_{\alpha_1} \times B_\beta$, if for any $n > 0$ and $i = 0, 1, \cdots, n-1$, we define $h = h_n = \alpha_1/n$ and

$$\begin{aligned} t_{i+1} = t_{i+1}^{(n)} = t_i + h, \qquad x_{i+1} = x_{i+1}^{(n)} = x_i + f(t_i, x_i)h, \\ \phi_n(t) = \phi_{h_n}(t) = x_i + f(t_i, x_i)(t - t_i) \qquad \forall t_i \le t \le t_{i+1}, \end{aligned}$$

then as $n \to \infty$,

$$\left| \phi_n(t) - x_0 - \int_{t_0}^{t} f(s, \phi_n(s))ds \right| \to 0 \qquad \forall t \in [t_0, t_0 + \alpha_1].$$

5. Recall that $\mathcal{C}(I, R^n)$ is the Banach space of all continuous functions from $I = [0,1]$ to $\mathbb{R}^n$ with the supremum norm. Prove that if the functions in a subset $E \subset C(I, R^n)$ are uniformly bounded and equicontinuous, then the closure of E is compact.

6. Reall the Implicit Function Theorem: Suppose that $F(x,y)$ is a continuously differentiable function from an open set $D \subset \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that $F(x_0, y_0) = 0$ and $(\partial F/\partial y)(x_0, y_0) \neq 0$ for some $(x_0, y_0) \in D$. Then there is an open set $I \subset \mathbb{R}$ containing x_0 and a unique differentiable function $\phi : I \to \mathbb{R}$ such that $F(x, \phi(x)) = 0$ for all $x \in I$.

(i) Prove that under the condition of the theorem, the function ϕ is unique.

(ii) Use the Existence Theorem of ODE to prove such ϕ exists.

7. Prove that any Lipschitz function ϕ from an open interval (a, b) to a Banach space X can be continuously extended to the endpoints a and b of the interval.

8. Suppose $f(t, x, \lambda)$ is a continuous function defined on an open set $D \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k$, and there are constants $M, K > 0$ such that

1) $|f(t, x, \lambda)| \leq M \quad \forall (t, x, \lambda) \in D$, and
2) $|f(t, x_1, \lambda) - f(t, x_2, \lambda)| \leq K|x_1 - x_2| \quad \forall (t, x_1, \lambda), (t, x_2, \lambda) \in D$.

Prove that for any $(t_0, x_0, \lambda_0) \in D$, there exists a number $\alpha > 0$ and a neighborhood V of (t_0, x_0, λ_0) such that $\forall (u, y, \lambda) \in V$, the initial value problem

$$\dot{x} = f(t, x, \lambda), \qquad x(u) = y,$$

has a unique solution $\phi(t, u, y, \lambda)$ defined on the interval $[u - \alpha, u + \alpha]$.

9. Let X be a Banach space, and let $E \subset X$. The *convex hull* of E, denoted by $\text{co}(E)$, is the intersection of all convex subsets of X which contain E.

(a) A *convex combination* of the points x_i is a point x of the form

$$x = \sum_{i=1}^{n} t_i x_i,$$

where each $t_i \in [0, 1]$ and $\sum_{i=1}^{n} t_i = 1$. Show that $\text{co}(E)$ coincides with the set of all convex combinations of elements of E.

(b) A metric space (X, d) is *totally bounded* if, for each $\varepsilon > 0$, there is a finite set of points $x_1, x_2, \ldots, x_n$ in X such that

$$X = \bigcup_{i=1}^{n} B_\varepsilon(x_i),$$

where $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$. That is, for every $\varepsilon > 0$, X can be covered by finitely many balls of radius ε. Show that if a subset $E \subset X$ is *totally bounded*, so are $\text{co}(E)$ and the closure of $\text{co}(E)$.

10. Recall that for a subset E of a Banach space X, its *closed convex hull*, denoted by $\overline{\text{co}}(E)$, is the intersection of all closed convex subsets of X which contain E.

(a) Prove that the closure of a convex set is also convex.

(b) Prove that the closure of $\text{co}(E)$ is $\overline{\text{co}}(E)$.

(c) Prove Mazur's theorem: the closed convex hull of a compact subset $E \subset X$ is also compact. (**Hint**: *Use the result in Exercise 9 and the fact that a subset in X is compact if and only if it is complete and totally bounded.*)