MATH 848 **Exercise Set 3** **Fall, 2014**

1. Suppose that $f(t,x,\lambda)$ is a C^1 function of the variables (t,x,λ) in an open set $D \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k$. For $(t_0,x_0,\lambda_0) \in D$, let $\phi(t,t_0,x_0,\lambda_0)$ be the solution of the initial value problem

$$\dot{x} = f(t,x,\lambda_0), \qquad x(t_0) = x_0. \tag{1}$$

Prove that the partial derivatives $\frac{\partial}{\partial t_0}\phi(t,t_0,x_0,\lambda_0)$ and of the solution with respect to the initial time t_0 satisfies the initial value problem

$$\dot{Z} = \frac{\partial f}{\partial x_0}(t,\phi(t,t_0,x_0,\lambda_0),\lambda_0)\cdot Z, \qquad \frac{\partial \phi}{\partial t_0}(t,t_0,x_0,\lambda_0)\Big|_{t=t_0} = -f(t_0,x_0,\lambda_0).$$

2. Let ϕ be the solution of the initial value problem.

(a) $\dot{x} = x + x^2 + tx^3$, $x(2) = x_0$. Find $\frac{\partial}{\partial x_0}\phi(t,2,x_0)|_{x_0=0}$.

(b) $\dot{x} = \frac{x}{t} + \lambda t e^{-x}$, $x(1) = 1$. Find $\frac{\partial}{\partial \lambda}\phi(t,1,1,\lambda)|_{\lambda=0}$.

3. Construct an autonomous polynomial vector field $X(x,y) = (f(x,y), g(x,y))$ (i.e., $f(x,y), g(x,y)$ are polynomials) in the plane such that X has a critical point at $(0,0)$, a periodic orbit γ which coincides the unit circle, and for each $z = (x,y) \neq (0,0)$, the $\omega-$limit set $\omega(z)$ equals γ. (**Hint**: *Find a suitable vector field in polar coordinates, and then change to (x,y) variables to get the desired vector field X.*)

4. Consider the differential equation $\dot{x} = f(x)$ where f is C^1 on $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and let $\phi(t,x)$ be the solution of $\dot{x} = f(x)$ such that $\phi(0,x) = x$. Prove that if $\phi(t,x)$ is defined and bounded for all $t \geq 0$, then the $\omega-$limit set $\omega(x)$ of x is a compact, non-empty, connected set.

5. In the situation of the preceding exercise, give an example to show that if the boundedness assumption of $\phi(t,x)$ fails, then $\phi(t,x)$ may exist for all $t \geq 0$, but $\omega(x)$ may be disconnected.

6. Recall that a point $x \in D$ is a *nonwanndering point* for a flow ϕ_t defined on a region $D \subset \mathbb{R}^n$ if for any neighborhood U of x and any $T > 0$, there exists $t > T$ such that $\phi_t(U) \cap U \neq \emptyset$.

Prove that a point x is a nonwanndering point for a flow ϕ_t if and only if for any neighborhood U of x, there exists $t \geq 1$ such that $\phi_t(U) \cap U \neq \emptyset$.

7. Suppose that $\rho : U \to V$ is a C^1 diffeomorphism from an open set $U \subset \mathbb{R}^n$ to $V \subset \mathbb{R}^n$. Let f and $g = \rho_*(f)$, defined by $g(y) = D\rho_x(f(x))$, $y = \rho(x)$, be the vector fields that define a flow on U and V respectively. Show that ρ maps

(a) the orbits of f passing through x to the orbits of g passing through y;

(b) $\alpha(x)$ and $\omega(x)$ to $\alpha(y)$ and $\omega(y)$ respectively;

(c) $\Omega(f)$ to $\Omega(g)$, where $\Omega(f)$ and $\Omega(g)$ are the nonwandering set of the flow of f and g respectively.

8. Let $f : D \to \mathbb{R}$ be a C^2 real-valued function defined in the open set $D \subset \mathbb{R}^n$. Let $\text{grad}(f)(x)$ be the gradient of f at the point $x \in D$; i.e., $\text{grad}(f)(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right)$. The system

$$\dot{x} = \text{grad}(f)(x) \tag{$*$}$$

is called a *gradient system* with potential funtion f. Let $\phi(t, x)$ be the local flow of the system $(*)$.

(a) Show that if x is not a critical point of $(*)$, then the function $f(\phi(t, x))$ is strictly increasing for t near 0.

(b) Suppose f is defined and C^2 on all of $\mathbb{R}^n$. Show that a solution $\phi(t, x)$ of $(*)$ is bounded if and only if $\omega(x)$ consists of a bounded set of critical points of the function f.

9. Determine, with justifiation, which of the following systems in $\mathbb{R}^2$ is a gradient system. If the system is a gradient system, determine the potential function f.

(a) $\dot{x} = y^2 - \sin(x), \quad \dot{y} = -y^2 + 2xy$

(b) $\dot{x} = y + \exp(x), \quad \dot{y} = x$

(c) $\dot{x} = y^3 - \sin(x), \quad \dot{y} = -x^2 + y$

10. Suppose $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ are solutions of the initial value problem

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1; \end{aligned} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

respectively. **Do not solve the equation** and prove the following:

(a) $\dot{\phi} = \psi$ and $\dot{\psi} = -\phi$;

(b) $\|\phi(t)\| = \|\phi(t)\| = 1$ for any $t \in \mathbb{R}$, where $\| \cdot \|$ denotes the Euclidian norm;

(c) both ϕ and ψ are periodic solutions with the same period;

(d) $\phi(t + \alpha) = \psi(t)$ and $\psi(t + \alpha) = -\phi(t)$ for any t if 4α is the period of ϕ.
(**Hint**: *Take α as the first zero of ϕ_2 on the positive x-axis.*)