MATH 848 **Exercise Set 4** **Fall, 2014**

1. Construct an autonomous polynomial vector field $X(x,y) = (f(x,y), g(x,y))$ (i.e., $f(x,y), g(x,y)$ are polynomials) in the plane such that X has a critical point at $p_0 = (0,0)$ and $p_1 = (1,0)$, a separatrix γ which coincides the unit circle $\mathbb{S}^1 \setminus p_1$, and for each $z = (x,y) \neq (0,0)$, the $\omega-$limit set $\omega(z)$ equals $\gamma \cup p_1$. (**Hint**: *you may use the result in Problem 3 of Homework 3, and the fact that* $\dot{x} = f(x)$ *and* $\dot{x} = \alpha(x)f(x)$ *have the same solution curves whenever* $\alpha(x) \neq 0$ *on the curves, where* $\alpha : \mathbb{R}^2 \to \mathbb{R}$ *is a local Lipschitz function.*)

2. Let $f = (P, Q)$ be a C^1 vector field from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(a) Prove that along any solution curve γ, the line integral

$$\int_\gamma P dy - Q dx = 0.$$

(b) Recall that $\mathrm{div}(f)(x)$ is the divergence of f at the point x; i.e., $\mathrm{div}(f)(x) = \frac{\partial P}{\partial x_1}(x) + \frac{\partial Q}{\partial x_2}(x)$. Prove the following Bendixson theorem: Suppose $\mathrm{div}(f) \geq 0$ on a simply connected region G, and $\mathrm{div}(f) \not\equiv 0$ on any simply connected subregion of G, then there is no closed orbit or separatrix cycle contained in G.

3. Show that according as $ad - bc > 0$ or $ad - bc < 0$, the index of the origin with respect to the linear vector field $f_0(x,y) = (ax + by, cx + dy)$ is ± 1.

4. Suppose that $f(x,y) = (f_1(x,y), f_2(x,y))$ is a C^1 vector field with an isolated critical point at $0 \in \mathbb{R}^2$ and the derivative of f at 0 is the linear map f_0 in Exercise 3. Show that if $ad - bc > 0$, then the index of f at 0 is $+1$ while if $ad - bc < 0$, then the index at 0 of f is -1.

5. Suppose that $\rho : U \to V$ is a C^1 diffeomorphism from an open set $U \subset \mathbb{R}^2$ to $V \subset \mathbb{R}^2$. Let f and $g = \rho_*(f)$, defined by $g(y) = D\rho_x(f(x))$, $y = \rho(x)$, be the vector fields on U and V respectively.

(a) Prove that if $x_0 \in U$ is an isolated critical point of f, then $y_0 = \rho(x_0)$ is an isolated critical point of g.

(b) Prove $\mathrm{Ind}(f, x_0) = \mathrm{Ind}(g, y_0)$ is $\det Df(x_0) \neq 0$.

(c) Is $\mathrm{Ind}(f, x_0) = \mathrm{Ind}(g, y_0)$ still true without the condition $\det Df(x_0) \neq 0$?

6. (a) Let $f(z) = z^k$ where $z = x + iy$ and z^k means the complex number z is multiplied by itself k-times. Consider f as a vector field in $\mathbb{R}^2$. Show that the index of f at 0 is k.

(b) Let $f(z) = \bar{z}^k$ where $z = x + iy$ and $\bar{z}^k$ means the complex conjugate of z multiplied by itself k times. Consider f as a vector field in $\mathbb{R}^2$. Show that the index of f at 0 is $-k$.

7. Show that the vector fields $f(z) = z^3$ and $g(z) = z^3 + z^4$ in $\mathbb{R}^2$ have the same index at $z = 0$.

8. Consider a differential equation of the form

$$\dot{x} = A(t)x + h(t)$$

where $A(t)$ is a continuous real $n \times n$ matrix valued function and $h(t)$ is a continuous real n-vector valued function. Assume that $h \neq 0$.

(a) Show that the equation has $n + 1$ independent solutions.

(b) Show that if $x_0(t), x_1(t), \cdots, x_n(t)$ are any $n + 1$ independent solutions of the equation, then the solution set of the equation is

$$\mathcal{S} = \left\{ \sum_{i=0}^{n} c_i x_i(t) : \ c_i \in \mathbb{R}, \ \sum_{i=0}^{n} c_i = 1 \ \right\}.$$

9. Show that if the linear differential equations

$$\dot{x} = A(t)x \qquad \text{and} \qquad \dot{x} = B(t)x$$

have the same fundamental matrix, then $A(t) = B(t)$.

10. Suppose that $\Phi(t)$ is a fundamental matrix of the linear differential equation $\dot{x} = A(t)x$. Show that a matrix $\Psi(t)$ is also a fundamental matrix of the equation if and only if there is a nonsingular constant matrix C such that $\Psi(t) = \Phi(t)C$.