MATH 848 **Exercise Set 5** **Fall, 2014**

1. Consider a linear system $\dot{x} = A(t)x$, where $A(t)$ is an $n \times n$ matrix valued real function defined on $\mathbb{R}$. Denote by $\phi(t, x)$ the solution satisfying the initial condition $x(0) = x$. Assume that $\phi(t, x)$ is bounded for any $x \in \mathbb{R}^n$, there is a number $B = B(x) > 0$ such that $|\phi(t, x)| \leq B \ \forall t \in [0, \infty)$. Prove that for any $x \in \mathbb{R}^n$, for any $\varepsilon > 0$, there exist $\delta > 0$, such that if $|x - y| < \delta$, then $|\phi(t, x) - \phi(t, y)| < \varepsilon$ for all $t \in [0, \infty)$. (**Hint**: *Prove the result for the zero solution* $\phi(t, 0) = 0$ *first.*)

2. Let $A(t)$ and $B(t)$ be $n \times n$ matrix valued real functions defined on $\mathbb{R}$. Let $\Phi(t)$ be the fundamental matrix of the equation $\dot{x} = A(t)x$ satisfying $\Phi(0) = I_n$, where I_n is the $n \times n$ identity matrix.

(a) Suppose that if there exist constants $a > 0$ and $\sigma > 0$ such that $\|\Phi(t)\| \leq ae^{-\sigma t}$. Prove that any solution $\phi(t)$ to the equation $\dot{x} = A(t)x$ satisfies $\lim\limits_{t\to\infty} \phi(t) = 0$, and the convergence is exponentially fast.

(b) Suppose in addition that the matrix $B(t)$ satisfies $\displaystyle\int_0^\infty \|B(s)\|ds < \infty$. Prove that any solution $\psi(t)$ to the equation $\dot{x} = (A(t)+B(t))x$ satisfies $\lim\limits_{t\to\infty} \psi(t) = 0$, and the convergence is exponentially fast.

3. Show that $\det e^A = \det e^B$ for $A = \begin{pmatrix} 1 & 0 & 3 \\ * & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & * \\ 2 & 1 & 5 \\ 0 & 1 & -2 \end{pmatrix}$, where the numbers at the position with $*$ are missing.

4. Find the general real solution to the differential equation $\dot{x} = Ax$ for each of the following matrices A. Sketch the orbits in two dimensional case.

(a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (c) $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$ (e) $A = \begin{pmatrix} 0 & 1 & 0 \\ 4 & 3 & -4 \\ 1 & 2 & -1 \end{pmatrix}$

5. Find a real-valued matrix C such that $A = C^2$, where A is the matrix

(a) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$; (b) $\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$; (c) $\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$.

6. Find a real-valued matrix B such that $A = e^B$, where A is the matrix

(a) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$; (b) $\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$; (c) $\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$.

7. Consider a differential equation of the form

$$\dot{x} = A(t)x + h(t),$$

where $A(t)$ is a continuous real $n \times n$ matrix valued function and $h(t)$ is a continuous real n-vector valued function, and satsify

$$A(t+T) = A(t), \qquad h(t+T) = h(t).$$

(a) Prove that a solution $\phi(t)$ to the equation is periodic with period T if and only if $\phi(T) = \phi(0)$.

(b) Prove that the equation has a unique periodic sulution with period T if and only if the matrix $Y(T)$ has no eigenvalue equal to 1, where $Y(t)$ is the fundanmental matrix to the corresponding homogeneous equation such that $Y(0) = I$.

8. Let X be a C^1 vector field of $\mathbb{R}^n$, and x_0 a critical point of X. Let $\phi_t(x)$ be the flow of the vector field X. Show that the critical point x_0 is hyperbolic if and only if the linear map $D\phi_t(x_0)$ is hyperbolic.

9. (Lipschitz Inverse Function Theorem). Let $(V, |\cdot|)$ be a Banach space, and suppose $f : V \to V$ is 1-1, onto, and Lipschitz with Lipschitz inverse. Prove that there is an $\epsilon > 0$ such that if $g = f + \phi$ where ϕ is Lipschitz with $|\phi|_0 < \epsilon$ and $Lip(\phi) < \epsilon$, then g is 1-1, onto, and Lipschitz with Lipschitz inverse.

10. Let $U, V \subset \mathbb{R}^n$ be open sets, and ϕ_t and ϕ_t are flows on U and V, respectively, that are defined for all $t \in \mathbb{R}$. Suppose that for any $t \in \mathbb{R}$, there is a unique diffeomorphism $h_t : U \to V$ such that $\phi_t \circ h_t = h_t \circ \psi_t$. Prove that h is independet of t.