MATH 848 **Exercise Set 6** **Fall, 2014**

1. Let

$$\dot{x} = Ax + f(t,x),$$

where A is a real constant matrix with eigenvalues λ_k all having negative real parts, and f is a real function continuous for small $|x|$ and $t \geq 0$, and

$$f(t,x) = o(|x|) \qquad\qquad (|x| \to 0)$$

uniformly in t, $t \geq 0$. You know that the solution $\phi(t) = 0$ is asymptotically stable.

Prove that if

$$\max_k \{\mathrm{Re}\,\lambda_k\} = -\alpha < 0,$$

then for any sulotion $\phi(t)$ of the equation which tands to 0 as $t \to +\infty$ satisfies

$$\limsup_{t\to+\infty} \frac{\log|\phi(t)|}{t} \leq -\alpha.$$

2. Suppose that $f(t)$ and $g(t)$ are continuous functions from $[0,+\infty)$ to $\mathbb{R}$. Show that every solution of the equation

$$\dot{x} = g(t)x + f(t)$$

(i) is stable if $\int_0^{+\infty} g(t)dt < +\infty$;

(ii) is asymptotically stable if $\int_0^{+\infty} g(t)dt = -\infty$;

(iii) is not stable if $\int_0^{+\infty} g(t)dt = +\infty$.

3. Recall that if $V : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function, then the *gradient vector field* with potential V is the vector field $\mathrm{grad}\,V$ defined by

$$(\mathrm{grad}\,V)(x) = \Big(\frac{\partial V}{\partial x_1}(x), \ldots, \frac{\partial V}{\partial x_n}(x)\Big).$$

Prove that V is a strict Lyapunov function for $\mathrm{grad}\,V$.

4. A critical point of the function V is a point x_0 such that $(\mathrm{grad}\,V)(x_0) = 0$. The critical point x_0 is *non-degenerate* if the Hessian matrix (a_{ij}), where

$$a_{ij} = \frac{\partial^2 V}{\partial x_i \partial x_j}(x_0),$$

is a non-singular matrix.

Prove that x_0 is a non-degenerate critical point of V if and only if it is a hyperbolic critical point of $\mathrm{grad}\,V$.

5. Find a suitable Lyapunov function and detrmine stability for the following systems.

(a) $\dot{x} = -xy^2, \;\; \dot{y} = -x^2 y$

(b) $\dot{x} = 2x^2 - 2y^2, \;\; \dot{y} = xy$

(c) $\dot{x} = -xy^6, \;\; \dot{y} = x^4 y^3$

(d) $\dot{x} = -y + \alpha x^3, \;\; \dot{y} = x + \alpha y^3$

6. Consider the second order differential equation

$$\ddot{x} + \dot{x} + \sin(2x) = 0.$$

Form the associated first order system, determine the critical points and determine stability of each. Sketch the orbits in the plane.

7. Consider the differential equations

$$\dot{x} = y, \qquad \dot{y} = -f(x,y)y - g(x),$$

where f and g are continuous functions such that $f(x,y) \geq 0$ for any $(x,y) \in \mathbb{R}^2$, and $xg(x) > 0$ for any $x \in \mathbb{R} \setminus \{0\}$. Determine stability of the zero solution.

8. Show that any critical point of a C^2 Hamiltonian system is not asymptotically stable.

9. Suppose a Hamiltonian function $H(x,y)$ defined on $\mathbb{R}^2$ satisfies

$$\lim_{x^2+y^2 \to \infty} |H(x,y)| = \infty.$$

Show that if the Hamiltonian system $\dot{x} = \dfrac{\partial H}{\partial y}$, $\dot{y} = -\dfrac{\partial H}{\partial y}$ has only finitely many critical points, then it has at least one Lyapunov stable critical point.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function and let $x_0 \in \mathbb{R}$ be a point at which $f(x_0) = 0$ and $f'(x_0) > 0$. Prove that the second order differential equation

$$\ddot{x} + f(x) = 0$$

has a nonconstant periodic solution.