1 Introduction

This course will cover basic material about ordinary differential equations. The main reference for the initial parts are the First 3 chapters of Hale, Ordinary Differential Equations, 2nd. ed.

Definition 1.1. *A* real normed linear (vector) space *is an ordered pair* $(\mathcal{X}, \|\cdot\|)$ *where* $\mathcal{X}$ *is a real vector space and* $|\cdot|\colon \mathcal{X} \to \mathbb{R}$ *is a real-valued function on* $\mathcal{X}$ *such that*

(i) $\|x\| \geq 0\ \forall x$ *and* $\|x\| = 0$ *iff* $x = 0$ *for* $x \in \mathcal{X}$*;*
(ii) $\|\alpha x\| = \|\alpha\|\|x\|$ *for* $\alpha \in \mathbb{R}, x \in \mathcal{X}$*;*
(iii) $\|x + y\| \leq \|x\| + \|y\|\ \forall\ x, y \in \mathcal{X}$.

If $\mathcal{X}$ *is a complex vector space and (ii) holds for all* $\alpha \in \mathbf{C}$*, then we call* $(\mathcal{X}, \|\cdot\|)$ *a* complex normed linear space.

Sometimes we say simply that $\mathcal{X}$ is a *normed linear space* where we understand that the norm $\|\cdot\|$ is given implicitly.

Definition 1.2. *A* metric space *is an ordered pair* $(\mathcal{X}, d)$*, where* $\mathcal{X}$ *is a set and* d *is a* metric *on* $\mathcal{X}$*. That is,* $d : \mathcal{X} \times \mathcal{X}$ *is a function such that for any* $x, y, z \in \mathcal{X}$*,*

(i) $d(x, y) \geq 0$ *for all* $x, y \in \mathcal{X}$*, and* $d(x, y) = 0$ *iff* $x = y$*;*
(ii) $d(x, y) = d(y, x)$*;*
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

The inequality in (iii) is called *triangle inequality.*

Fact. *If* $(\mathcal{X}, \|\cdot\|)$ *is a normed linear space, then* $d(x, y) = \|x - y\|$ *induces a metric in* $\mathcal{X}$.

We say that a sequence $(x_1, x_2, \ldots)$ in $(\mathcal{X}, d)$ is a *Cauchy sequence* if, for every $\epsilon > 0$, there is an $N > 0$ such that

$$n, m \geq N \Longrightarrow d(x_n, x_m) < \epsilon$$

The metric space $(\mathcal{X}, d)$ is *complete* if every Cauchy sequence in $\mathcal{X}$ converges to a point of $\mathcal{X}$.

Fact. *Every closed subset of a complete metric space is again complete.*

Definition 1.3. *A normed linear space* $(\mathcal{X}, \|\cdot\|)$ *is called a* Banach space *if it is a complete metric space with respect to the metric* $d(x, y) = \|x - y\|$ *induced by the norm.*

In other words, a Banach space is a complete normed vector space with the metric induced by the norm.

Let us give some examples of normed linear spaces and Banach spaces.

Examples.

1. Let $\mathcal{X} = R^n$ or $\mathcal{X} = \mathbf{C}^n$ denote the sets of $n-$tuples of real and complex numbers, respectively. Define the following norms $|\cdot|$ in $\mathcal{X}$.

(a) $|x|_p = (\sum_{1\leq i\leq n} |x_i|^p)^{\frac{1}{p}}$;
(b) $|x|_\infty = \max_{1\leq i\leq n} |x_i|$,

where $x = (x_1, \ldots, x_n) \in \mathcal{X}$.

With any one of these norms, $\mathcal{X}$ becomes a Banach space. The *usual* norm is $|\cdot|_2$ above.

It is instructive to consider the geometric pictures of the unit balls in each of the above Banach Spaces.

2. A linear subspace V of a Banach space $\mathcal{X}$ is itself a Banach space if and only if it is closed.

3. Let D be a compact subset of $\mathbb{R}^n$. The set $\mathcal{C}(D, \mathbb{R}^n)$ of continuous functions from D to $\mathbb{R}^n$ becomes a Banach Space with the norm

$$\|f\| = \sup_{x\in D} |f(x)|.$$

4. Let X, Y be Banach spaces. Let $B(X, Y)$ be the set of bounded functions from X to Y with the *sup* norm

$$\|f\| = \sup_{x\in X} |fx|.$$

Then, $(B(X, Y), \|\cdot\|)$ is itself a Banach space.

The set of bounded continuous functions $BC(X, Y)$ with the sup norm is a closed subspace of $B(X, Y)$.

A function $F : X \to Y$ between metric spaces is called *Lipschitz* if there is a constant $L > 0$ such that

$$d(Fx, Fy) \leq Ld(x, y)$$

for all $x, y \in X$. The smallest such constant,

$$\sup_{x\neq y\in X} \frac{d(Fx, Fy)}{d(x, y)}$$

is called the *Lipschitz constant* of F.

Let X, Y be Banach spaces, let $L > 0$, and let $\mathcal{L}_L(X, Y)$ be the set of bounded functions from X to Y which are Lipschitz with Lipschitz constant less than or equal to L. Then, with the *sup* norm, $\mathcal{L}_L(X, Y)$ is a closed subset of $BC(X, Y)$.

It can be proved that in Example 1, $|x|_\infty = \lim_{p\to\infty} |x|_p$ for any $x \in \mathcal{X}$.

The norm given by Example 3 and 4 are called *uniform norm*, or *supremum norm*, or *infinity norm.* It is called the uniform norm since any sequence $\{f_n\} \subset \mathcal{C}(D, \mathbb{R}^n)$ converges to $f \in \mathcal{C}(D, \mathbb{R}^n)$ under the metric if and only if the convergence $f_n \to f$ is uniform.

Exercises:

(1) Suppose that E is a finite dimensional linear vector space. Let $|\cdot|_1, |\cdot|_2$ be two norms on E. Show that there are constants $C_1, C_2 > 0$ such that, for all $x \in E$,

$$C_1|x|_1 \le |x|_2 \le C_2|x|_1$$

(2) Let I be the real unit interval, and let $\mathcal{C}(I, \mathbb{R})$ be the space of continuous real-valued functions on I with the L^1 *norm*

$$\|f\| = \int_I |f(x)|\,dx$$

Show that this makes $(\mathcal{C}(I, \mathbb{R}), |\cdot|)$ a normed linear space, but that it is not complete. What is the completion of this space?

Note that for a metric space D, $\mathcal{C}(D, \mathbb{R}^n)$ denote the Banach space consisting of all continuous function from D to $\mathbb{R}^n$ with the uniform norm.

Definition 1.4. *Let D be a compact metric space, and $\mathcal{F} \subset \mathcal{C}(D, \mathbb{R}^n)$. We say that the family $\mathcal{F}$ of functions is* equicontinuous *if for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$x, y \in D \text{ and } d(x, y) < \delta \longrightarrow \|fx - fy\| < \epsilon \quad \forall\, f \in \mathcal{F}.$$

We say the family $\mathcal{F}$ is uniformly bounded *if there is a constant $C > 0$ such that $\|fx\| < C$ for all $f \in \mathcal{F}, x \in \mathcal{X}$.*

Theorem (Arzela-Ascoli Theorem)**.** *Let D be a compact metric space. A closed subset $\mathcal{F} \subset \mathcal{C}(D, \mathbb{R}^n)$ is compact in the uniform topology if and only if it is closed, bounded and equicontinuous.*

Note that a subset $\mathcal{S}$ in a metric space is compact if and only if any sequence in $\mathcal{S}$ contains a convergent subsequence. So by the thoerem, we see that

> *if a sequence $\{f_n\} \subset \mathcal{C}(D, \mathbb{R}^n)$ is uniformly bounded and equicontinuous, then it has a convergent subsequence.*

Example. Let D be a compact metric space. For $K, L > 0$, let $\mathcal{L}_{L,K}(D, \mathbb{R}^n)$ be the space of Lipschitz functions from D to $\mathbb{R}^n$ with norm less than or equal to K and Lipschitz constant less than or equal to L. Clearly, $\mathcal{C}(D, \mathbb{R}^n)$ is a closed subset of $\mathcal{C}(D, \mathbb{R}^n)$. By Arzela-Ascoli theorem, $\mathcal{L}_{L,K}(D, \mathbb{R}^n)$ is a compact. In particular, every sequence in $\mathcal{L}_{L,K}(D, \mathbb{R}^n)$ has a subsequence which converges to an element of $\mathcal{L}_{L,K}(D, \mathbb{R}^n)$.