# 12. Linear Differential Equations with Constant Coefficients

We now consider differential equations of the form

$$\dot{x} = Ax \tag{12.1}$$

with $A$ an $n \times n$ real or complex matrix.

If $n = 1$, then we know the general solution has the form

$$x(t) = e^{At}x_0.$$

So, it is tempting to try to obtain a similar formula for the matrix case.

Consider the matrix power series

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots. \tag{12.2}$$

The inequality $| AB | \leq | A || B |$ for $n \times n$ matrices implies $| A^k | \leq | A |^k$. Hence the series (12.2) converges to a unique matrix. We call this matrix $e^A$ or $\exp(A)$.

It is easy to see that if $A$ and $B$ commute (i.e., $AB = BA$), then $e^{A+B} = e^A \cdot e^B$.

Then, we can see that the matrix function $e^{tA}$ defines a smooth function of $t$ and, for any constant vector $x_0$, the function

$$x(t) = e^{tA}x_0 \tag{12.3}$$

solves the initial value problem $\dot{x} = Ax, x(0) = x_0$.

The matrix $e^{tA}$ is a fundamental matrix for (12.1), since its columns consist of solutions (with initial values $e_j$, the standard unit vectors), and its determinant is nowhere zero.

The form (12.3) of the solution to (12.1) is useful for many purposes, but in some contexts it is useful to have other forms for the solutions to (12.1).

Suppose that the matrix $A$ is diagonalizable, that is, there exists a nonsingular matrix $X$ and $n$ numbers $\lambda_1, \dots, \lambda_n$ such that

$$D = X^{-1}AX, \qquad \text{or} \qquad AX = XD, \tag{12.4}$$

where $D = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$ is a diagonal matrix.

Denote $X = (x_1, \dots, x_n)$, where $x_j$ is the $j$th column vector of $X$. The above equation gives that

$$A(x_1, \dots, x_n) = (x_1, \dots, x_n)D = (\lambda_1 x_1, \dots, \lambda_n x_n).$$

That is, the elements in the diagonal of $D$ are the eigenvalues of $A$, and the column vectors of $X$ are the corresponding eigenvectors of $A$.

The first equation in (12.4) also gives

$$e^{tD} = X^{-1}e^{tA}X, \qquad \text{or} \qquad e^{tA}X = Xe^{tD}.$$

Take the $j$th column of the last equality we have that

$$\phi_j(t) := e^{tA}x_j = e^{\lambda_j t}x_j$$

is a solutions of (12.1). Hence, $(\phi_1(t), \ldots, \phi_n(t))$ form a form a fundamental set of solutions since the eigenvectors $x_1, \ldots, x_n$ are linearly independent. Thus, the general solution has the form

$$x(t) = \sum_j \alpha_j e^{t\lambda_j} x_j.$$

Now we suppose that the eigenvalues of $A$ are all real, but that they are not necessarily distinct. Hence, $A$ is not necessary diagonalizable.

Write the characteristic polynomial of $A$, $p(\lambda) = \det(\lambda I - A)$ as

$$p(\lambda) = (\lambda - \lambda_1)^{m_1} \ldots (\lambda - \lambda_s)^{m_s}$$

for some $m_1, \ldots, m_s$. The real numbers $\lambda_1, \ldots, \lambda_s$ are the distinct eigenvalues of $A$, and $\sum_j m_j = n$.

Denote

$$V_j = \ker(A - \lambda_j I)^{m_j}.$$

From linear algebra, we know that

$$p(A) = (A - \lambda_1 I)^{m_1} \ldots (A - \lambda_s I)^{m_s} = 0.$$

That is, for any vector $v \in \mathbb{R}^n$, $p(A)v = 0$. Hence, there is a direct sum decomposition

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \ldots \oplus V_s.$$

Also we have the following.

**Fact 12.1.** *$V_j$ have the following properties.*

1. *$A(V_j) \subseteq V_j$, that is, $V_j$ is invariant under $A$.*
2. *$\dim V_j = m_j$.*
3. *For any $1 \le j \le s$, there is an integer $0 < r_j \le m_j$ such that for each $v \in V_j$, $(A - \lambda_j I)^{r_j}(v) = 0$, and there is a $v \in V_j$ such that $(A - \lambda_j I)^{r_j - 1}v \ne 0$.*
4. *$V_j$ consists of only eigenvalues if and only if $r_j = 1$.*

It is easy to see part (1), because if $v \in V_j$, then

$$(A - \lambda_j I)^{m_j} Av = A(A - \lambda_j I)^{m_j} v = A0 = 0,$$

hance $Av \in V_j$. Also, part (3) and (4) are obvious.

The subspace $V_j$ is called the *generalized eigenspace* of $A$ corresponding to the eigenvalue $\lambda_j$.

In the case that the matrix $A$ is diagonalizable, each $V_j$ is the eigenspace of $\lambda_j$ and $r_j = 1$. In particular, if all the eigenvalues are distinct, then $\dim V_j = 1$ and $r_j = 1$ for any $1 \le j \le n$.

To find the general solution of (12.1), it suffices to find $n$ linearly independent solutions. For this purpose, it suffices to work in each generalized eigenspace $V_j$.

For each $j$, denote $N_j = A - \lambda_j I$. Then $V_j$ is the set of vectors $v$ such that there is a positive integer $1 \leq r = r_j \leq m_j$ such that $N_j^r v = 0$. Note that $V_j \supsetneq \{0\}$ since the eigenvectors corresponding to $\lambda_j$ are in $V_j$.

**Lemma 12.2.** *For any $v \in V_j$ with $N^{r_j-1}v \neq 0$, $v, N_j v, \ldots, N^{r_j-1}v$ are linearly independent.*

*Proof.* Suppose

$$c_0 v + c_1 N_j v + \ldots + c_{r_j-1} N_j^{r_j-1} v = 0.$$

Apply $N_j^{r_j-1}$ we get $c_0 N^{r_j-1} v = 0$. Since $N_j^{r_j-1} v \neq 0$, we have $c_0 = 0$. Hence we have

$$c_1 N_j v + \ldots + c_{r_j-1} N_j^{r_j-1} v = 0.$$

Apply $N_j^{r_j-2}, \ldots, N_j$, we get $c_1 = \ldots = c_{r_j-2} = 0$, and then $c_{r_j-1}$. □

**Definition 12.1.** *A subspace $W$ of a vector space is called a* cyclic subspace *for a linear transformation $N$ if there are a vector $w \in W$ and a positive integer $r$ such that $N^{r-1}w \neq 0$, $N^r w = 0$, and $W =$* span$(w, Nw, N^2 w, \ldots, N^{r-1}w)$.

*In this case, $w$ is called a* cyclic generator *of $W$ of* order $a$.

We also call $W$ a cyclic subspace of order $r$.

**Remark.**
(1) *A cyclic subspace $W$ is invariant under $N$. That is, $Nv \in W$ for any $v \in W$.*
(2) *If $w$ is a cyclic generator of $W$ of order $r$, then the vectors $w, Nw, \ldots, N^{r-1}w$ form a basis for $W$. This is because by Lemma 12.2, the set of vectors form a linearly independent set.*

**Definition 12.2.** *An operator $N$ is called* nilpotent *on a vector space $V$, if $N^a = 0$ for some integer $r > 0$. That is, there is a positive integer $m$ such that $N^r v = 0$ for all $v \in V_j$.*

We will use the following theorem in algebra.

**Theorem 12.3.** *If $N$ is a nilpotent operator on a finite dimensional vector space $V$, then $V$ is a direct sum of cyclic subspaces. More precisely, there are cyclic subspaces $W_1, W_2, \ldots, W_k$ and positive integers $r_1, \ldots, r_k$ such that $W_i$ is cyclic of order $r_i$ and*

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_k.$$

Let us apply this theorem to the subspaces $V_j$ and operators $N_j = A - \lambda_j I$ above. The operator $N_j$ is nilpotent on $V_j$. So we may write

$$V_j = W_{j1} \oplus W_{j2} \oplus \ldots \oplus W_{jk},$$

where each $W_{ji}$ is cyclic of order $r_i$. Let $w_i$ be a generator for $N_j$ on $W_{ji}$. Then, the set $\{N_j^l w_i : 0 \leq l < r_i,\ 1 \leq i \leq k\}$ is a basis for $V_j$.

Next, let us indicate how we can obtain the general solution to (12.1) in a single cyclic subspace $W_{ji}$ of $N_j$.

To simplify the notation, assume that $W = W_{ji},\ m = r_i = dimW,\ \lambda = \lambda_j,\ N = N_j|_W$. Then, $A|_W$ has only the eigenvalue $\lambda$, this eigenvalue has multiplicity $m$, and $N = A - \lambda I$. The vector $w = w_i$ is in $W$, and the vectors $w, Nw, N^2 w, \ldots, N^{m-1} w$ form a basis for $W$.

**Lemma 12.4.** *The functions*

$$x_{m-j}(t) = e^{t\lambda} \left( \sum_{k=m-j}^{m-1} \frac{t^{k-m+j}}{(k-m+j)!} N^k w \right)$$

*for* $j = 1, \ldots, m$ *form a fundamental set of solutions of of* $\dot{x} = Ax$ *on* $W$*.*

Before giving the proof, let us write the solutions

$$x_0(t), x_1(t), \ldots, x_{m-1}(t)$$

in a more extended form:

$$\begin{aligned} x_{m-1}(t) &= e^{\lambda t} N^{m-1} w, \\ x_{m-2}(t) &= e^{\lambda t} \left( N^{m-2} w + t N^{m-1} w \right), \\ x_{m-3}(t) &= e^{\lambda t} \left( N^{m-3} w + t N^{m-2} w + \frac{t^2}{2!} N^{m-1} w \right), \\ &\vdots \\ x_0(t) &= e^{\lambda t} \left( w + tNw + \ldots + \frac{t^{m-1}}{(m-1)!} N^{m-1} w \right). \end{aligned}$$

*Proof of Lemma 12.4.* The vectors $w, Nw, \ldots, N^{m-1} w$ are linearly independent in $W$.

If we show the functions $x_{m-j}(t)$ are solutions, then it follows that they are linearly independent since their values at $t = 0$ are the independent vectors $w, Nw, \ldots, N^{m-1} w$.

But, the function $x_{m-j}(t)$ has the form

$$x_{m-j}(t) = e^{t(\lambda I + N)} N^{m-j} w = e^{tA} N^{m-j} w,$$

which is a solution of (12.1). $\square$

**Remark 12.5.** *In general to apply the above results one needs to first find a cyclic vector $w$ of order $r$ for the subspace $W$. If $\lambda$ is an eigenvalue of multiplicity $k$ of $A$, and* $\ker(A - \lambda I)$ *is one dimensional, then there is a simple procedure to find a cyclic vector for $A - \lambda I$ on $W$. Let $w_0$ be any eigenvector for $A$ associated to $\lambda$ which is in the image of $A - \lambda I$, and in general, if $w_{i-1}$ is in* $\ker(A - \lambda I)^i$ *for some $i < k$, let $w_i$ be such that $(A - \lambda I)w_i = w_{i-1}$. Then, $w_{k-1}$ is a cyclic vector for $A - \lambda I$ on $W$, and $\{w_{k-1}, \ldots w_1, w_0\}$ form a basis for $W$.*

Now, suppose that $A$ has some complex eigenvalues, but is a real matrix.

If $\mu$ is a complex eigenvalue of $A$, and the corresponding eigenvector is $w$, where $w$ is a vector in $\mathbb{C}^n$, then the equation $\dot{z} = Az$ has complex solutions

$$z(t) = e^{\mu t} w \qquad \text{and} \qquad \overline{z}(t) = e^{\overline{\mu} t} \overline{w}.$$

Hence, the equation $\dot{x} = Ax$ has solutions

$$\mathrm{Re}(e^{\mu t} w) \qquad \text{and} \qquad \mathrm{Im}(e^{\mu t} w).$$

In general, if $\mu$ is a complex eigenvalue of multiplicity $k$, then we can decompose $W = \ker(A - \mu I)^k$ into cyclic subspaces and get basis $(w, Nw, \ldots N^{r-1}w)$ for each cyclic subspace, where $r$ is the dimension of the cyclic subspace. Then we can get solutions $z_{r-1}(t), \ldots, z_1(t), z_0(t)$. The real parts and imaginary parts of the solutions are the solutions to the equation $\dot{x} = Ax$.

**Examples.**

**1.**

$$\begin{aligned} x' &= -2x \\ y' &= y \end{aligned}$$

Here the matrix $A$ is $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$, the eigenvalues are $-2, 1$, and the general solution is

$$\bar{x}(t) = e^{-2t} \begin{pmatrix} c_1 \\ 0 \end{pmatrix} + e^{t} \begin{pmatrix} 0 \\ c_2 \end{pmatrix}.$$

The critical point 0 is called a *saddle*.

**2.**

$$\begin{aligned} x' &= 2x - y \\ y' &= x + y \end{aligned}$$

The matrix $A$ is $\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. The characteristic polynomial is $\lambda^2 - 3\lambda + 3$, and the eigenvalues are $\lambda = \dfrac{3}{2} \pm i\dfrac{\sqrt{3}}{2}$.

Letting $\lambda = \dfrac{3}{2} + i\dfrac{\sqrt{3}}{2}$, we have the matrix equation

$$(A - \lambda I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives $(2-\lambda)v_1 = v_2$, so that a complex eigenvalue is $(v_1, v_2) = (1, 2-\lambda)$.

We get a complex solution of the form

$$\begin{aligned} \bar{x}_c(t) =& e^{\lambda t}\begin{pmatrix} 1 \\ 2-\lambda \end{pmatrix} = e^{(\frac{3}{2}+i\frac{\sqrt{3}}{2}t)}\begin{pmatrix} 1 \\ \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix} \\ =& e^{(\frac{3}{2}+i\frac{\sqrt{3}}{2}t)}\left(\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + i\begin{bmatrix} 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix}\right). \end{aligned}$$

The real and imaginary parts of this are

$$\begin{aligned} \mathrm{Re} =& e^{\frac{3}{2}t}\left(\cos(\frac{\sqrt{3}}{2}t)\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} - \sin(\frac{\sqrt{3}}{2}t)\begin{bmatrix} 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix}\right), \\ \mathrm{Im} =& e^{\frac{3}{2}t}\left(\cos(\frac{\sqrt{3}}{2}t)\begin{bmatrix} 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix} + \sin(\frac{\sqrt{3}}{2}t)\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}\right). \end{aligned}$$

**3.**

$$\begin{aligned} x' &= 2x \\ y' &= 2y \end{aligned}$$

The matrix $A$ is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The characteristic polynomial is $(\lambda - 2)^2$, and the only eigenvalue is 2.

The general solution is

$$x(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

**4.**

$$\begin{aligned} x' &= x + y \\ y' &= y \end{aligned}$$

The matrix is $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We have

$$e^{tA} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \left[ I + t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}.$$

So the general solution is

$$\mathbf{x}(t) = e^{tA} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} c_1 e^t + c_2 te^t \\ c_2 e^t \end{bmatrix}.$$

**5.**

$$\begin{aligned} x' &= 3x + 11y + 5z \\ y' &= -x - y - z \\ z' &= 2x + z \end{aligned}$$

The matrix is $A = \begin{pmatrix} 3 & 11 & 5 \\ -1 & -1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$. The characteristic polynomial is

$$p(\lambda) = \lambda^3 - 3\lambda^2 + 4 = (\lambda - 2)^2(\lambda + 1).$$

*The eigenvalue* $\lambda = 2$:

Let $N = A - 2I$. Then, $N = \begin{bmatrix} 1 & 11 & 5 \\ -1 & -3 & -1 \\ 2 & 0 & -1 \end{bmatrix}$ with $\operatorname{rank}(N) = 2$, and $N^2 = \begin{bmatrix} 0 & -22 & -11 \\ 0 & -2 & -1 \\ 0 & 22 & 11 \end{bmatrix}$ with $\operatorname{rank}(N)^2 = 1$.

The vector $v = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ is in $\ker(N)$. The vector $w = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ satisfies $Nw = v$. We get two linearly independent solutions in $\ker(N^2)$ by

$$e^{2t}v, \quad e^{2t}(w + tNw).$$

*The eigenvalue* $\lambda = -1$:

Let $N = A + I$. Then, $N = \begin{bmatrix} 4 & 11 & 5 \\ -1 & 0 & -1 \\ 2 & 0 & 2 \end{bmatrix}$, $\operatorname{rank}(N) = 2$, and $\ker(N)$ is one-dimensional.

The vector $v = \begin{pmatrix} -1 \\ -\frac{1}{11} \\ 1 \end{pmatrix}$ is in the kernel of $N$, so is an eigenvector for $A$ associated to $\lambda = -1$.

A fundamental set of solutions, then, is the set

$$e^{2t}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \; e^{2t}(I+tN)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; e^{-t}\begin{pmatrix} -1 \\ -\frac{1}{11} \\ 1 \end{pmatrix},$$

where

$$N = \begin{bmatrix} 1 & 11 & 5 \\ -1 & -3 & -1 \\ 2 & 0 & -1 \end{bmatrix}.$$

## Appendix: Jordan canonical form

Recall that the characteristic polynomial of an $n \times n$ matrix $A$, $p(\lambda) = \det(\lambda I - A)$, can be written as

$$p(\lambda) = (\lambda - \lambda_1)^{m_1} \dots (\lambda - \lambda_s)^{m_s},$$

for some $m_1, \dots, m_s$ with $\sum_j m_j = n$, where the real numbers $\lambda_1, \dots, \lambda_s$ are the distinct eigenvalues of $A$.

If we denote $V_j = \ker(A - \lambda_j I)^{m_j}$. then there is a direct sum decomposition

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \dots \oplus V_s,$$

where $\dim V_j = m_j$. For each $j$, $1 \le j \le s$, $V_j$ is a direct sum of cyclic subspace of $\mathbb{R}^n$

$$V_j = W_{j1} \oplus W_{j2} \oplus \dots \oplus W_{jt_j},$$

where each $W_{ji}$ is cyclic of order $r_{ji}$ with respect to the operator $N_j = A - \lambda_j I$, and $\sum_i r_{ji} = m_j$.

Moreover, on each $W_{ji}$, there exists a vector $w = w_{ji}$, a cyclic generator, such that the set

$$\{N_j^{r_{ji}-1}w, \; \dots, \; N_j w, \; w\} \tag{12.5}$$

form a basis for $W_{ji}$. Hence, if we take this basis, the matrix representation of the operator is a Jordan block

$$B_{ji} = \begin{pmatrix} \lambda_j & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_j & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & \lambda_j & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_j \end{pmatrix} \tag{12.6}$$

restricted to $W_{ji}$. So if we take basis of the form (12.5) for each $W_{ji}$, then restricted to $V_j$, the matrix has the form

$$J_j = \begin{pmatrix} B_{j1} & 0 & \dots & 0 & 0 \\ 0 & B_{j2} & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 0 & B_{jt_j} \end{pmatrix}. \tag{12.7}$$

A Jordan form of the matrix $A$ is

$$J = \operatorname{diag}(J_1, \dots, J_s).$$

If a matrix $A$ has complex eigenvalues $\lambda_j$, then the Jordan form is a complex matrix. Now we consider the case that $A$ is an $n \times n$ real matrix and it has complex eigenvalues $\lambda_j$.

The complex eigenvalues come in complex conjugate pairs

$$\mu_1, \overline{\mu}_1, \mu_2, \overline{\mu}_2, \dots, \mu_k, \overline{\mu}_k,$$

where $\mu_j = \lambda_j + i\kappa_j, \overline{\mu}_j = \lambda_j - i\kappa_j$, and $i = \sqrt{-1}$.

If $\mu = \lambda + i\kappa$ is a complex eigenvalue of $A$, with corresponding eigenvector $w = u + iv$, then $\overline{w} = u - iv$ is a eigenvector of $A$ belongs to the eigenvalue $\mu = \lambda - i\kappa$. That is,

$$A(u + iv) = (\lambda + i\kappa)(u + iv), \qquad A(u - iv) = (\lambda - i\kappa)(u - iv).$$

Hence we get that

$$\begin{aligned} Au &= \operatorname{Re} A(u + iv) = \operatorname{Re}(\lambda + i\kappa)(u + iv) = \lambda u - \kappa v, \\ Av &= \operatorname{Im} A(u + iv) = \operatorname{Im}(\lambda + i\kappa)(u + iv) = \kappa u + \lambda v. \end{aligned}$$

So if we take $(u, v)$ as a basis in the subspace $W = \operatorname{span}(v, u)$ spanned by $u$ and $v$, then restricted to $W$, $A$ has the form

$$\begin{pmatrix} \lambda & -\kappa \\ \kappa & \lambda \end{pmatrix}. \tag{12.8}$$

In general, if $W$ is a cyclic subspace for $N = A - \mu I$, where $\mu = \lambda + i\kappa$ is a complex eigenvalue of $A$, then there is a cyclic subspace $\overline{W}$ for $\overline{N} = A - \overline{\mu} I$ with $r = \dim N = \dim \overline{N}$. In this case, if $w = u + iv \in W$ is a cyclic generator of $W$ for the nilpotent operator $N$, then $\overline{w} = u - iv \in \overline{W}$ is a cyclic generator of $\overline{W}$ for $\overline{N}$. Hence we have $A(N^k w) = \mu N^k w + N^{k+1} w$ for $k = 1, \dots r - 2$ and $A(N^{r-1} w) = \mu N^{r-1} w + N^r w = \mu N^{r-1} w$, and therefore

$$\begin{aligned} A(\operatorname{Re} N^k w) &= \operatorname{Re} A(N^k w) = \lambda \operatorname{Re} N^k w - \kappa \operatorname{Im} N^k w + \operatorname{Re} N^{k+1} w, \\ A(\operatorname{Im} N^k w) &= \operatorname{Im} A(N^k w) = \kappa \operatorname{Re} N^k w + \kappa \operatorname{Im} N^k w + \operatorname{Im} N^{k+1} w, \end{aligned}$$

for $k = 1, \dots r-2$ and

$$A(\operatorname{Re} N^{r-1}w) = \operatorname{Re} A(N^{r-1}w) = \lambda \operatorname{Re} N^{r-1}w - \kappa \operatorname{Im} N^{r-1}w,$$
$$A(\operatorname{Im} N^{r-1}w) = \operatorname{Im} A(N^{r-1}w) = \kappa \operatorname{Re} N^{r-1}w + \kappa \operatorname{Im} N^{r-1}w.$$

So if we take

$$\{\operatorname{Re} N^{r-1}w,\ \operatorname{Im} N^{r-1}w,\ \dots,\ \operatorname{Re} Nw,\ \operatorname{Im} Nw,\ \operatorname{Re} w,\ \operatorname{Im} v\}$$

as a basis for $W \oplus \overline{W}$, then restricted to $W \oplus \overline{W}$, the Jordan block has the form

$$B_{ji} = \begin{pmatrix} M & I & 0 & \dots & 0 & 0 \\ 0 & M & I & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & M & I \\ 0 & 0 & 0 & \dots & 0 & M \end{pmatrix}, \tag{12.9}$$

where $M = \begin{pmatrix} \lambda & -\kappa \\ \kappa & \lambda \end{pmatrix}$ and $I = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$.