13. Linear Periodic Systems

Consider the homogeneous linear periodic system

$$\dot{x} = A(t)x, \qquad A(t+T) = A(t), \qquad \forall t \in \mathbb{R}, \tag{13.1}$$

where $T > 0$ and $A(t)$ is a continuous $n \times n$ real or complex matrix which is periodic of period T in t.

We will prove that there is a nonsingular periodic transformation of variables of period T or $2T$ taking (**??**) into a linear differential equation with constant coefficients.

We begin with some properties of e^A for a matrix A. We assume that all matrices we mention below are $n \times n$ matrices.

Fact 13.1. *(1) $e^a \cdot e^B = e^{A+B}$ if $AB = BA$.*

(2) $P^{-1}e^A P = e^{P^{-1}AP}$ for any nonsingular matrix P.

Proof. The proof can be obtained directly from the definition. □

Recall that a matrix N is nilpotent if $N^k = 0$ for some $k \geq 1$.

Lemma 13.2. *Suppose $A = C + N$ for some nonsingular matrix C and nilpotent matrix N such that $CN = NC$. If there exists a matrix D such that $C = e^D$, then there exists a matrix B such that $A = e^B$.*

Proof. Since C is nonsingular, we can write

$$A = C + N = C(I + C^{-1}N). \tag{13.2}$$

Since $CN = NC$, if $N^k = 0$, then $(C^{-1}N)^k = C^{-k}N^k = 0$.

Note that if $s = \log(1+a) = a - \frac{1}{2}a^2 + \frac{1}{3}a^3 + \ldots$, then $1 + s = e^a$. So we take

$$S = C^{-1}N - \frac{1}{2}(C^{-1}N)^2 + \ldots + \frac{(-1)^{n-1}}{n-1}(C^{-1}N)^{(n-1)}. \tag{13.3}$$

Then it is easy to check

$$I + C^{-1}N = e^S.$$

Hence, by Fact 13.1 (1), (13.2) implies

$$S = e^D \cdot e^S = e^{D+S}.$$

So the result follows with $B = D + S$. □

Lemma 13.3. *(a) If A is an $n \times n$ complex matrix with $\det A \neq 0$, then there is a complex matrix B such that $A = e^B$.*

(b) If A is an $n \times n$ real matrix, then there is a real matrix B such that $A = e^B$ if and only if $\det A \neq 0$ and A is a square, that is, there is a real matrix C such that $A = C^2$.

Proof. By Fact 13.1 (2), we only need to consider the case that A is of a Jordan form J. This is because if $A = P^{-1}JP$ for some nonsingular matrix P, then $J = e^B$ implies $A = e^{P^{-1}BP}$.

(a) Let J be a complex Jordan form of A, and Λ be the diagonal matrix that has the same diagonal elements with J. Since A is nonsingular, so are J and Λ.

Let $N = J - \Lambda$. Clearly, N is a nilpotent matrix and $\Lambda N = N\Lambda$.

Since Λ is a nonsingular diagonal matrix, the elements on the diagonal are all nonzero. Hence, it is easy to see that there is a complex diagonal matrix D such that $\Lambda = e^D$. By the above lemma we get that $J = E^B$ for some matrix B.

(b) The necessity is easy since we simply take $C = e^{\frac{1}{2}B}$, and observe that $\det(e^B) \neq 0$ for any B if $A = e^B$.

For sufficiency, we take a real Jordan form of A, that is, if λ is a real eigenvalue of A, then the corresponding Jordan has the form as in (12.6) and (12.7), and if $\mu = \lambda + i\kappa$, $\lambda, \kappa \in \mathbb{R}$, is a complex eigenvalue of A, then the corresponding Jordan has the form as in (12.9) and (12.7).

Then we take Λ such that on each Jordan block Λ has the form λI if the block is corresponding a real eigenvalue λ of A, and the form $\text{diag}(M, \ldots, M)$ if the block is corresponding to a complex eigenvalue $\mu = \lambda + i\kappa$ of A where M is a 2×2 matrix as in (12.8). Then denote $N = J - \Lambda$. Clearly N is a nilpotent matrix and $\Lambda N = N\Lambda$.

If $\lambda > 0$, then $\lambda I = e^{(\log \lambda)I}$.

If $\lambda < 0$, then restricted to the general eiganspace $V = \ker(J - \lambda I)^k$, where k is the algebraic multiplicity of λ. By abusing the notations we can regard a matrix as a linear transformation. We have $\det J|_V = \det A|_V = \det C^2|_V > 0$. So the dimension of V must be even and λ has even multiplicity $k = 2m$. Note that there exists a 2×2 matrix D_1 such that $e^{D_1} = \lambda I_2$, where I_2 is a 2×2 matrix. Hence if we take the $k \times k$ matrix $B_1 = \text{diag}\{D_1, \ldots, D_1\}$, we have $e^{B_1} = \lambda I_k$.

If $\lambda = \lambda + i\kappa$ is a complex number, then on the corresponding Jordan block, Λ has the form $\text{diag}\{M, \ldots, M\}$, where M is a 2×2 matrix given in (12.8). Since we can find a 2×2 matrix D_2 such that $e^{D_2} = M$, we have that $e^{B_2} = \text{diag}\{M, \ldots, M\}$, where $B_2 = \text{diag}\{D_2, \ldots, D_2\}$.

Hence we get that there is an $n \times n$ matrix D such that $e^D = \lambda$. By Lemma 13.2, we obtain the result. □

Theorem 13.4 (Floquet)**.** *Every fundamental matrix $\Phi(t)$ for (13.1) has the form*

$$\Phi(t) = P(t)e^{Bt} \tag{13.4}$$

where $P(t)$ is a periodic matrix of period T and B is a constant matrix (which may by complex). We may always obtain (13.1) with a real matrix B where $P(t)$ has period $2T$.

Proof. Let $\Phi(t)$ be a fundamental matrix for (13.1).

Then, letting $u = u(t) = t + T$, and using $A(t+T) = A(t)$, we get

$$\begin{aligned}\frac{d}{dt}\Phi(t+T) =& \frac{d}{du}\Phi(u) = A(u)\Phi(u) \\ =& A(t+T)\Phi(t+T) = A(t)\Phi(t+T).\end{aligned}$$

So, $\Phi(t+T)$ is also a solution matrix. Since it is nonsingular, it is a fundamental matrix. Thus, there is a nonsingular matrix A such that

$$\Phi(t+T) = \Phi(t)A, \tag{13.5}$$

and A is a real matrix if $\Phi(t)$ is real. By Lemma 13.3, there is a (possibly complex) matrix B such that $e^{BT} = A$.

Now, letting $P(t) = \Phi(t)e^{-Bt}$ we get $\Phi(t) = P(t)e^{Bt}$ and

$$P(t+T) = \Phi(t+T)e^{-B(t+T)} = \Phi(t)e^{-Bt} = P(t).$$

In order to choose B to be real, we simply need the matrix A to be a square of some real matrix. But by (13.5), we have

$$\Phi(t+2T) = \Phi(t+T+T) = \Phi(t+T)A = \Phi(t)A^2.$$

Thus, replacing T by $2T$ in (13.5), we may obtain a real matrix B such that $e^{2TB} = A^2$. Repeating the above argument then gives the result. □

Corollary 13.5. *There is a nonsingular periodic transformation of variables of period T or $2T$ taking (13.1) into a linear differential equation with constant coefficients.*

Proof. Let $P(t)$, B be as above, and set $x = P(t)y$.

We may choose $P(t)$ to be of period T or $2T$ as above.

Then,

$$\dot{x} = \dot{P}y + P\dot{y} = Ax = APy.$$

So,

$$APy = \dot{P}y + P\dot{y}.$$

But, $P = \Phi e^{-Bt}$, or $Pe^{Bt} = \Phi$, so

$$\dot{P}e^{Bt} + PBe^{Bt} = APe^{Bt},$$

or

$$\dot{P} + PB = AP,$$

or

$$APy = (AP - PB)y + P\dot{y},$$

or

$$PBy = P\dot{y},$$

or $By = \dot{y}$ since P is non-singular. $\square$