The Grobman-Hartman theorem

Now that we have studied the structure of solutions to linear differential equations in general, we wish to use that theory to study the local structure of the solutions to non-linear systems.

If X is a C^r vector field, $r \geq 1$, defined in an open set $U \subset \mathbb{R}^n$, and $x_0 \in U$ is a non-singular point (i.e., $X(x_0) \neq 0$), then we have seen that there is a C^r change of coordinates which takes solutions near x_0 to straight lines. Thus, it remains to describe the solutions near a critical point. If the derivative $A = DX_{x_0}$ of X at x_0 has eigenvalues with real parts different from zero, we will see, that after a continuous change of coordinates, the structure of solutions of X near x_0 is the same as that of the linear system $\dot{y} = Ay$ near 0.

14.1. **Definitions and Statements of the Theorems.** We now make the relevant definitions.

Let X be a C^r vector field as above with $r \geq 1$ with a critical point at x_0 (i.e., $X(x_0) = 0$). Let A be the derivative of X at x_0. Thus, $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map whose matrix in the standard coordinates on $\mathbb{R}^n$ is the Jacobian matrix of X at x_0.

Definition 14.1. *The critical point x_0 of X is called* hyperbolic *if the eigenvalues of $A = DX_{x_0}$ all have non-zero real parts.*

Recall that the *local flow* of of a C^r, $r \geq 1$, vector field X near x_0 is the function $\eta(t, x)$ defined in a neighborhood V of $(0, x_0)$ in $\mathbb{R}^{n+1}$ such that

(1) $\eta(0, x) = x$ for $(0, x) \in V$;
(2) $t \to \eta(t, x)$ is a solution to the differential equation $\dot{x} = X(x)$ defined in a neighborhood of $t = 0$.

We also use the notation η_t for the local flow $\eta(t, x)$. We sometimes call η_t the local flow of the differential equation $\dot{x} = X(x)$ as well. We will also use the term *integral curve* of the vector field X for a solution curve.

Definition 14.2. *A linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ is called* hyperbolic *if its (possibly complex) eigenvalues have norm different from one.*

Example. (1) *The map L induced by the 2×2 matrix* $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.
(2) $L = e^A$, *where A is a linear map whose eigenvalues have non-zero real parts.*

If L is a hyperbolic linear map of $\mathbb{R}^n$, then there is a direct sum decomposition $\mathbb{R}^n = E^s \oplus E^u$ such that

(1) $L(E^s) = E^s$ and $L(E^u) = E^u$;
(2) the eigenvalues of $L|_{E^s}$ have norm less than 1 and those of $L|_{E^u}$ have norm greater than 1.

In fact, we can take $E^s = \oplus_{|\lambda_i|<1} V_i$ and $E^u = \oplus_{|\lambda_i|>1} V_i$, where V_i be the general eigenspace corresponding eigenvalue λ_i, i.e. $V_i = \ker(L-\lambda_i I)^{m_i}$ and m_i is the algebraic multiplicity of λ_i.

Definition 14.3. *Let X be a C^r vector field defined in a neighborhood of x_0 in $\mathbb{R}^n$ having x_0 as a critical point. Let DX_{x_0} be the derivative of X at x_0. A C^0* linearization *of X near x_0 is a homeomorphism h from a neighborhood U of x_0 in $\mathbb{R}^n$ to a neighborhood of 0 such that if η_t is the local flow of X near x_0, then $h\eta_t h^{-1}$ is the local flow of the linear differential equation $\dot{y} = DX_{x_0} \cdot y$ near 0.*

One may similarly define C^k linearizations of a C^r vector field X for $1 \leq k \leq r$ by requiring that h be a C^k diffeomorphism from a neighborhood of x_0 to a neighborhood of 0.

Theorem 14.1. *[Grobman-Hartman] Suppose x_0 is a hyperbolic critical point of the C^1 vector field X. Then X has a C^0 linearization near x_0.*

For smooth linearizations, one has the following result.

Theorem. *Suppose that L is linear map on $\mathbb{R}^n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of L. For each positive integer k, there is a positive integer $N(k)$ with the following property. Suppose that for each $1 \leq i \leq n$ and each n-tuple $(m_1, m_2, \ldots, m_n)$ of non-negative integers satisfying $2 \leq \sum_{1\leq j\leq n} m_j \leq N(k)$, we have $\lambda_i \neq \sum_{1\leq j\leq n} m_j \lambda_j$.*

Then, any $C^{N(k)}$ vector field X with $X(x_0) = 0$ and $DX_{x_0} = L$ has a local C^k linearization near x_0.

Note that as a corollary of the Grobman-Hartman theorem, we can get that for a hyperbolic critical point x_0 of a C^1 vector field X in $\mathbb{R}^n$, if all the eigenvalues of the derivative $L = DX_{x_0}$ have negative real parts, then x_0 is asymptotically stable. If L has at least one eigenvalue with positive real part, then x_0 is unstable. The statement and the meaning of stablility are in the next section.

We will proceed toward the proof of Theorem14.1. Note that we may assume that both X and L have local flows defined for $| t | \leq 1$.

In the course of the proof, it will be necessary to first linearize the time-one map η_1 of X near x_0. So, we first study the relevant linearization theorem for local diffeomorphisms.

Definition 14.4. *Let f be a C^1 diffeomorphism from a neighborhood U of x_0 in $\mathbb{R}^n$ into $\mathbb{R}^n$ with $f(x_0) = x_0$. The fixed point x_0 is called* hyperbolic *if all the eigenvalues of* Df_{x_0} *have absolute values with norm different from one; i.e,* Df_0 *is a hyperbolic linear map.*

Fact. *If a critical point* x_0 *of a vecter field* f *is hyperbolic, and* ϕ_t *is the local flow, then* $D\phi_t(x_0)$ *is hyperbolic for any* $t \in \mathbb{R} \setminus \{0\}$.

Theorem 14.2. *[Grobman-Hartman theorem for local diffeomorphisms] Suppose* x_0 *is a hyperbolic fixed point of the local* C^1 *diffeomorphism* f *defined on a neighborhood* U *of* x_0 *in* $\mathbb{R}^n$*. Let* $L = Df_{x_0}$*. There is a neighborhood* $U_1 \subseteq U$ *of* x_0 *and a homeomorphism* h *from* U_1 *into* $\mathbb{R}^n$ *such that* $h(x_0) = 0$ *and* $hf(x) = Lh(x)$ *for* $x \in U_1 \bigcap f^{-1}U_1$.

Note that an equivalent formulation of

$$hf(x) = Lh(x) \quad \text{for } x \in U_1 \bigcap f^{-1}U_1$$

is

$$hfh^{-1}(y) = L(y) \quad \text{for } h^{-1}(y) \in U_1 \bigcap f^{-1}U_1$$

so the formulas in both theorems are analogous.

Remark.
(1) *The proofs we will give of the above theorems are valid if* $\mathbb{R}^n$ *is replaced by a Banach space.*
(2) *A map* h *as in Theorem 2 is called a* C^0 *linearization of* f*. One may define* C^k *linearizations analogously for* $k \geq 1$.

Let U be an open subset of $\mathbb{R}^n$ and let $g : U \to R^n$ be a mapping. Recal that g is *Lipschitz* (or *Lipschitz continuous*) if there is a constant $K > 0$ such that

$$\sup_{x \neq y} \frac{\mid g(x) - g(y) \mid}{\mid x - y \mid} \leq K < \infty.$$

When g is Lipschitz, the *Lipschitz constant* of g is given by $\text{Lip}(g) = \sup_{x \neq y} \frac{\mid g(x) - g(y) \mid}{\mid x - y \mid}$.

Note that if g is C^1 and $M = \sup_x \mid D_x g \mid$, then g is Lipschitz and $\text{Lip}(g) = M$. That is, the maximum of the norms of the derivatives of a C^1 map g equals the Lipschitz constant of g.

We will develop some machinery to prove Theorem 14.2. Then we will give a proof of Theorem 14.1.

Let us first note that, replacing $f(x)$ by $f(x + x_0) - x_0$, we may assume $x_0 = 0$.

Proposition 14.3. *Suppose* $L : \mathbb{R}^n \to \mathbb{R}^n$ *is a hyperbolic linear map. There is an* $\varepsilon > 0$ *depending on* L *such that the following holds.*

If $\phi_1, \phi_2 : \mathbb{R}^n \to \mathbb{R}^n$ *are Lipschitz maps such that*

$$|| \phi_i ||_0 \leq \varepsilon, \qquad Lip(\phi_i) < \varepsilon, \tag{14.1}$$

then there is a unique continuous map $h = h_{\phi_1 \phi_2} : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $h - \mathrm{id}$ *is a bounded continuous map and*

$$(L + \phi_1) \circ h = h \circ (L + \phi_2).$$

Let us first assume the proposition and prove Theorems 14.2 and 14.1.

Proof of Theorems 14.2. Take $L = D_0 f$, and then take $\varepsilon > 0$ as in Proposition 14.3. Then take $\delta > 0$ as in Propostion 14.10, and suppose f itself satisfies the conclusion (1)-(3) in the propostion. Denote $\phi = f - L$. Then ϕ satisfies (14.1). Let ζ denote the zero map, i.e. $\zeta(x) = 0$ for all x.

By Propostion 14.3, there is a unique map $h = h_{\zeta\phi}$ of bounded distance from the identity such that

$$Lh = hf.$$

To prove that h is a homeomorphism, we apply Propostion 14.3 again to get a unique map $h' = h_{\phi\zeta}$ of bounded distance from the identity such that

$$fh' = h'L.$$

This gives us $fh'h = h'Lh = h'hf$. So, $h'h$ is a continuous map and $h'h - \mathrm{id}$ is bounded. By uniqueness of the solution to $fh = hf$, we have $h'h = \mathrm{id}$. Similarly, $hh'L = Lh'h$, and by uniqueness of the solutions to $hL = Lh$, we have $hh' = \mathrm{id}$.

Thus, h is a homeomorphism and Theorem 14.2 is proved with the neighborhood $U_1 = B_{\delta/2}(x_0)$. □

Proof of Theorems 14.1. (It is left as an exercise.) □

14.2. **Norms for Linear Maps.** Before we prove Proposition 14.3, we introduce some lemmas concerning properties of linear maps.

Lemma 14.4. *Suppose* $H : V \to V$ *is a bounded linear self-map of the Banach space* V *with* $\mid H \mid < 1$. *Let* I *denote the identity map,* $Ix = x$. *Then,* $I - H$ *is an isomorphism and*

$$\mid (I - H)^{-1} \mid \leq \frac{1}{1 - \mid H \mid} \tag{14.2}$$

Proof. Let $T = \sum_{i=0}^{\infty} H^i$. Then, T is a bounded linear operator, and

$$(I - H)T = T(I - H) = I.$$

Therefore, $I - H$ is an isomorphism with inverse T.

Moreover,

$$\mid (I - H)^{-1} \mid = \mid T \mid \leq \sum_{i=0}^{\infty} \mid H \mid^i = \frac{1}{1 - \mid H \mid}.$$

This is what we need. □

Lemma 14.5. *If $V = V_1 \oplus V_2$ is a direct sum decomposition of the Banach space V, and $H : V \to V$ is an isomorphism such that $H(V_i) = V_i$ for $i = 1, 2$, $|| H \mid V_1 || < 1$, and $|| H^{-1} \mid V_2 || < 1$, then $I - H$ is an isomorphism.*

If V is given the maximum norm, then

$$| (I - H)^{-1} | \leq \max \left(\frac{1}{1 - |H|V_1|}, \frac{|H^{-1}|V_2|}{1 - |H^{-1}|V_2|} \right). \tag{14.3}$$

Proof. For $u = u_1 + u_2$ with $u_i \in V_i$, define

$$T(u) = T(u_1 + u_2) = \sum_{i=0}^{\infty} H^i(u_1) + \Big(- \sum_{i=1}^{\infty} H^{-i}(u_2) \Big).$$

Then, $(I - H)T = T(I - H) = I$. □

Lemma 14.6. *Suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map all of whose eigenvalues have norm less than one. Let $\tau_1 = \sup\{| \lambda |: \lambda$ is an eigenvalue of $L\}$. Let $\tau \in (\tau_1, 1)$. Then there is a new norm $|| \cdot ||$ on $\mathbb{R}^n$ such that $|| L(v) || \leq \tau || v ||$ for all $v \in \mathbb{R}^n$. That is, with respect to the norm $|| \cdot ||$ on L induced by the norm $|| \cdot ||$ on $\mathbb{R}^n$, we have $|| L || < \tau$.*

Proof. Using the fact that $L = S + N$ where S is semi-simple (complex diagonalizable) and N is nilpotent, one sees that there is a constant $C > 0$ such that $m \geq 0$ implies that $| L^m v | \leq C\tau^m \mid v \mid$ for all $v \in \mathbb{R}^n$. Thus, for each v, the quantity $\alpha(v) = \sup\{|L^m v|\tau^{-m} : m \geq 0\}$ is finite. Set $\|v\| = \alpha(v)$. Then it is easy to see that $|| \cdot ||$ is a norm on $\mathbb{R}^n$.

On the other hand,

$$\begin{aligned} || Lv || &= \sup(| L^m Lv \mid \tau^{-m} : m \geq 0) \\ &= \tau\tau^{-1} \sup(| L^m Lv \mid \tau^{-m} : m \geq 0\}) \\ &= \tau \sup(\{| L^m Lv \mid \tau^{-m-1} : m \geq 0\}) \\ &= \tau \sup(\{| L^{m+1} v \mid \tau^{-m-1} : m \geq 0\}) \\ &= \tau \sup(\{| L^m v \mid \tau^{-m} : m \geq 1\}) \leq \tau || v ||. \end{aligned}$$

This gives $|| L || < \tau$. □

Remark. *If we were dealing with a Banach space E instead of $\mathbb{R}^n$, we would just let τ_1 be the spectral radius of the operator L above.*

Lemma 14.7. *Suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear hyperbolic isomorphism. That is, no eigenvalues of L have norm 1. Let $\tau \in (0, 1)$ be such that the eigenvalues of L inside the unit circle have norm less than τ, and those outside the unit circle have norm great then τ^{-1}.*

Then, there is a direct sum decomposition $\mathbb{R}^n = V_1 \oplus V_2$ *and a new norm* $||\cdot||$ *on* $\mathbb{R}^n$ *such that*

$$L(V_1) = V_1, \quad L(V_2) = V_2, \tag{14.4}$$

and the norm induces a norm $||\cdot||$ *such that*

$$\|L \mid V_1\| < \tau, \quad \|L^{-1} \mid V_2\| < \tau. \tag{14.5}$$

Proof. Let $\mathbb{R}^n = V_1 \oplus V_2$ be the direct sum decomposition such that $L|V_1$ has eigenvalues less than τ in norm, and $L|V_2$ has eigenvalues greater than τ^{-1} in norm. Note that $L^{-1}|V_2$ has eigenvalues of norm less than τ. By Lemma 3, there are norms $||\cdot||_1$ and $||\cdot||_2$ on V_1 and V_2, respectively, such that (14.5) holds. For $v = (v_1, v_2)$ with $v_i \in V_i$, let $||\, v\, || = \max(||\, v_1\, ||, ||\, v_2\, ||)$. □

14.3. **Proof of the Main Proposition.**

Proof of Proposition 14.3. We consider that h has the form $h = \mathrm{id} + u$. For $\phi_1, \phi_2 : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathrm{Lip}(\phi_i) < \varepsilon$, we show that the equation

$$(L + \phi_1) \circ (\mathrm{id} + u) = (\mathrm{id} + u) \circ (L + \phi_2) \tag{14.6}$$

has a unique solution $u \in C^0_b(\mathbb{R}^n, \mathbb{R}^n) := \{u \in C^0(\mathbb{R}^n, \mathbb{R}^n)$: u is bounded$\}$ Equation (14.6) is equivalent to

$$L \circ \mathrm{id} + L \circ u + \phi_1 \circ (\mathrm{id} + u) = L + \phi_2 + u_1 \circ (L + \phi_2)$$

or

$$u - L^{-1} u \circ (L + \phi_2) = L^{-1}\phi_2 - L^{-1}\phi_1 \circ (\mathrm{id} + u). \tag{14.7}$$

Let $H : C^0_b(\mathbb{R}^n, \mathbb{R}^n) \to C^0_b(\mathbb{R}^n, \mathbb{R}^n)$ be defined by

$$H(v) = L^{-1} \circ v \circ (L + \phi_2),$$

and let $H_1 = I - H$ with I the identity transformation of $C^0(\mathbb{R}^n, \mathbb{R}^n)$. Then, both H and H_1 are bounded linear maps, and equation (14.7) becomes

$$H_1(u) = L^{-1}\phi_2 - L^{-1}\phi_1 \circ (\mathrm{id} + u_1). \tag{14.8}$$

By Lemma 14.8 below, we can write (14.7) as

$$\begin{aligned} u =& H_1^{-1}(L^{-1}\phi_2 - L^{-1}\phi_1 \circ (\mathrm{id} + u)) \\ =& H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (\mathrm{id} + u)) \end{aligned}$$

which means we want a fixed point in $C^0_b(\mathbb{R}^n, \mathbb{R}^n)$ of the map

$$T : v \to H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (\mathrm{id} + v))$$

We show that T is a contraction if ε is small. In fact,

$$\begin{aligned} \| Tu - Tv \|_0 &= \| H_1^{-1}(L^{-1}\phi_1 \circ (\mathrm{id}+v)) - H_1^{-1}(L^{-1}\phi_1 \circ (\mathrm{id}+u)) \|_0 \\ &\le \| H_1^{-1} \| \| L^{-1} \| \| \phi_1 \circ (\mathrm{id}+u) - \phi_1 \circ (\mathrm{id}+v) \|_0 \\ &\le \| H_1^{-1} \| \| L^{-1} \| (\mathrm{Lip}(\phi_1)) \| u - v \|_0. \end{aligned}$$

So, if

$$\mathrm{Lip}(\phi_1) \| L^{-1} \| \frac{1}{1-\tau} < 1,$$

then T is a contraction.

This completes the proof of Proposition 14.3. $\square$

Lemma 14.8. *H_1 is an isomorphism and* $\| H_1^{-1} \| \le \dfrac{1}{(1-\tau)}$.

Proof. Note that by the Lipschitz Inverse Function Theorem below, for ε small, $(L+\phi_2)^{-1}$ exists and is Lipschitz. This gives that H is an isomorphism with inverse $v \to L \circ v \circ (L+\phi_2)^{-1}$.

Let $\bar{V}_i = C_b^0(\mathbb{R}^n, V_i)$ for $i = 1, 2$. Then we have $C_b^0(\mathbb{R}^n, \mathbb{R}^n) = \bar{V}_1 \oplus \bar{V}_2$, $H(\bar{V}_i) = \bar{V}_i$, $i = 1, 2$, $\|H|\bar{V}_2\| < \tau$, and $\|H^{-1}|\bar{V}_1\| < \tau$. Thus, H is hyperbolic on $C_b^0(\mathbb{R}^n, \mathbb{R}^n)$. By Lemma 14.5 we have that H_1 is an isomorphism and $\| H_1 \| \le \dfrac{1}{1-\tau}$. $\square$

Theorem (Lipschitz Inverse Function Theorem)**.** *Let $(V, |\cdot|)$ be a Banach space, and suppose $f : V \to V$ is 1-1, onto, and Lipschitz with Lipschitz inverse. There is an $\varepsilon > 0$ such that if $g = f + \phi$ where ϕ is Lipschitz with $\| \phi \|_0 < \varepsilon$ and* $\mathrm{Lip}(\phi) < \varepsilon$, *then g is 1-1, onto, and Lipschitz with Lipschitz inverse.*

Proof. (It is left as an exercise.) $\square$

14.4. **Appendix: Bump Functions.**

Lemma 14.9. *There is a C^∞ function $\alpha : \mathbb{R} \to [0,1]$ such that*

(1) $\alpha(u) = 1$ *for* $u \le \frac{1}{2}$;
(2) $\alpha(u) = 0$ *for* $u \ge 1$.

Proof. Let

$$\phi(u) = \begin{cases} \exp\left(-\frac{1}{(\frac{1}{2}-u)(u-1)}\right) & \text{for } \frac{1}{2} < u < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then ϕ is C^∞.

Let

$$\psi(u) = \frac{\int_{-\infty}^{u} \phi(s)\, ds}{\int_{-\infty}^{1} \phi(s)\, ds}.$$

Then ψ is C^∞, and

$$\psi(u) = \begin{cases} 0 & \text{for } u \leq \frac{1}{2}; \\ 1 & \text{for } u \geq 1, \end{cases}$$

and $\psi(u) \in [0,1]$ for all u.

Let $\alpha(u) = 1 - \psi(u)$. Then, α has the required properties. (Details left as an exercise.) □

Let f be as in the statement of Theorem14.2. Our next lemma will show that we may assume there is a $\delta > 0$ such that f is defined on all of $\mathbb{R}^n$, $f(x) = L(x)$ for $|\, x \,| \geq \delta$ and $\mathrm{Lip}(f - L)$ (on all of $\mathbb{R}^n$) is small.

Proposition 14.10. *Suppose x_0 is a fixed point of the local C^1 diffeomorphism f defined on a neighborhood U of x_0 in $\mathbb{R}^n$ with $L = Df_{x_0}$. For any $\varepsilon > 0$, there are a $\delta > 0$ and a C^1 diffeomorphism $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ such that*

(1) $f_1(x) = L(x)$ *for* $|\, x \,| \geq \delta$
(2) $f_1(x) = f(x)$ *for* $|\, x \,| \leq \frac{\delta}{2}$
(3) $\mathrm{Lip}(f_1 - L) < \varepsilon$ *and* $\|\, f_1 - L \,\|_0 < \varepsilon$.

Here, $\|\, f_1 - L \,\|_0 = \sup_{x \in \mathbb{R}^n} |\, f_1(x) - L(x) \,|$.

Proof. Let $\varepsilon_1 \in (0,1)$, and let $\delta_1 \in (0,1)$ be small enough so that

(a) f is defined for $|\, x \,| \leq \delta_1$;
(b) $\|\, D_x(f - L) \,\| < \varepsilon_1$; and
(c) $|\, f(x) - L(x) \,| < \varepsilon_1 \delta_1$ for $|\, x \,| \leq \delta_1$.

Let α be as in Lemma 14.9, and let $K = \sup_{u \in \mathbb{R}} |\, \alpha'(u) \,|$.

Let $\gamma(x) = \alpha(\frac{|x|}{\delta_1})$. Note that $|\, D_x \gamma \,| \leq \frac{K}{\delta_1}$ for all x.

Now,

$$\gamma(x) = \begin{cases} 1 & \text{for } |\, x \,| \leq \frac{\delta_1}{2}; \\ \\ 0 & \text{for } |\, x \,| \geq \delta_1. \end{cases}$$

Let

$$\begin{aligned} f_1(x) &= \gamma(x) f(x) + (1 - \gamma(x)) L(x) \\ &= L(x) + \gamma(x)(f(x) - L(x)). \end{aligned}$$

Note that f_1 is the γ-average of f and L.

Now, $(f_1 - L)(x) = \gamma(x)(f(x) - L(x))$, so

$$\begin{aligned} \|\, f_1 - L \,\|_0 &= \sup_x |\, \gamma(x)(f(x) - L(x)) \,| \\ &\leq \sup_{|x| \leq \delta_1} |\, f(x) - L(x) \,| \leq \quad \varepsilon_1 \end{aligned}$$

Also,

$$\begin{aligned} \| D_x(f_1 - L) \| &= | D_x\gamma \cdot (f(x) - L(x)) + \gamma(x)(D_x f - L) | \\ &\leq \| D_x\gamma \| \, | f(x) - L(x) | + \| D_x f - L \|_0 \\ &\leq \frac{K}{\delta_1}\varepsilon_1\delta_1 + \varepsilon_1 \end{aligned}$$

Note that we use the notation $D_x\gamma \cdot (f(x) - L(x))$ for the map $v \to D_x\gamma(v)(f(x) - L(x))$.

Now, given $\varepsilon \in (0, 1)$, choose $\varepsilon_1 \in (0, 1)$ small enough so that $\max(\varepsilon_1, K\varepsilon_1 + \varepsilon_1) < \varepsilon$. □