2. Linear Transformations and Fixed Point Theorems

2.1. Linear Transformations.

Definition 2.1. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. A mapping $f : \mathcal{X} \to \mathcal{Y}$ is* linear *if it satisfies the following two properties:*

(1) *$f(x + y) = f(x) + f(y)$ for all $x, y \in \mathcal{X}$*
(2) *$f(\alpha x) = \alpha f(x)$ for all $x \in \mathcal{X}, \alpha \in \mathbb{R}$.*

A linear map is also called a *linear transformation*.

Definition 2.2. *A linear map $f : \mathcal{X} \to \mathcal{Y}$ is called* bounded *if there is a constant $C > 0$ such that $|f(x)| \leq C|x|$ for all $x \in \mathcal{X}$.*

Fact 2.1. *Linear maps have the following properties.*

(1) *A linear map is bounded if and only if it is continuous.*
(2) *The linear map f is bounded if and only if $\sup\limits_{|x|\leq 1} |f(x)|$ is finite.*
(3) *The quantity $\sup\limits_{|x|\leq 1} |f(x)|$ is also equal to $\sup\limits_{|x|=1} |f(x)|$.*
(4) *Every linear map whose domain is finitely dimensional linear space is bounded (hence continuous).*

If f is a bounded linear map (transformation), we set $|f| = \sup\limits_{|x|=1} |f(x)|$. This defines a norm in the space $L(\mathcal{X}, \mathcal{Y})$ of bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$, making it into a Banach space also.
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2.2. Fixed Point Theorems.

Many existence theorems for differential equations can be reduced to fixed point theorems in appropriate function spaces. Here we will discuss a few relevant results.

Let $\mathcal{X}$ be a metric space and let $T : \mathcal{X} \to \mathcal{X}$ be a mapping. A *fixed point* of T is a point $x \in \mathcal{X}$ such that $T(x) = x$.

A self-map T of a metric space $\mathcal{X}$ is called a *contraction* (or *contraction map*) if there is a constant $0 < \lambda < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) \quad \forall x, y \in \mathcal{X}.$$

Thus, $T : \mathcal{X} \to \mathcal{X}$ is a contraction if and only it is Lipschitz with Lipschitz constant less than 1.

Theorem 2.2 (Contraction Mapping Theorem)**.** *Suppose $\mathcal{X}$ is a complete metric space and $T : \mathcal{X} \to \mathcal{X}$ is a contraction map. Then, T has a unique fixed point $\bar{x}$ in $\mathcal{X}$.*

Moreover, if x is any point in $\mathcal{F}$, then the sequence of iterates x, Tx, $T^2x, \ldots$ converges to $\bar{x}$ exponentially fast.

Proof. (Uniqueness) If $0 < \lambda < 1$ is the contraction constant for T and $Tx = x, Ty = y$, then

$$d(x, y) = d(Tx, Ty) \leq \lambda d(x, y)$$

which implies that $d(x, y) = 0$. This in turn implies that $x = y$.

(Existence) Take any $x \in \mathcal{X}$ and let $x_0 = x$, $x_i = T^i x$ for $i > 0$. Then,

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \leq \ldots \leq \lambda^n d(x_1, x_0) \quad \forall n \geq 1.$$

Thus, for $m > n$,

$$\begin{aligned} d(x_m, x_n) \leq& d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\ \leq& (\lambda^{m-1} + \lambda^{m-2} + \ldots + \lambda^n) d(x_1, x_0) \\ =& \frac{\lambda^n(1 - \lambda^{m-n})}{1 - \lambda} d(x_1, x_0) \\ \leq& C\lambda^n d(x_1, x_0), \end{aligned} \tag{2.1}$$

where $C = 1/(1 - \lambda)$.

This implies that the sequence $\{x_i\}_{i=1,2,\ldots}$ is a Cauchy sequence. By completeness of $\mathcal{X}$, it converges, say to an element $\bar{x}$ of $\mathcal{X}$. But, since T is continuous,

$$T(\bar{x}) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = \bar{x},$$

so, $T(\bar{x}) = \bar{x}$. This proves the existence.

Since (2.1) holds for any $m \geq n$, let $m \to \infty$ we get

$$d(\bar{x}, x_n) \leq C\lambda^n d(x_1, x_0).$$

The fact $\lambda \in (0, 1)$ gives that the convergence is exponential. □

The preceding theorem gives a useful sufficient condition for the existence of fixed points in a wide variety of situations. It is frequently useful to know when such fixed points depend continuously on parameters. This leads us to the next result.

Definition 2.3. *Let Λ be a topological space (e.g. a metric space), and let $\mathcal{X}$ be a complete metric space. A map T from Λ into the space of maps $\mathcal{M}(\mathcal{X}, \mathcal{X})$ is called a* continuous family of self-maps of $\mathcal{X}$ *if the map $\bar{T}(\lambda, x) = T(\lambda)(x)$ is continuous as a map from the product space $\Lambda \times \mathcal{X}$ to $\mathcal{X}$.*

The map T is called a uniform family of contractions *on $\mathcal{X}$ if it is a continuous family of self-maps of $\mathcal{X}$ and there is a constant $0 < \alpha < 1$ such that*

$$d(\bar{T}(\lambda, x), \bar{T}(\lambda, y)) \leq \alpha d(x, y)$$

for all $x, y \in \mathcal{X}, \lambda \in \Lambda$.

Thus, the continuous family is a uniform family of contractions if and only if all the maps in the family have the same upper bound $\alpha < 1$ for their Lipschitz constants.

Given the family T as above, we define the map $T_\lambda : \mathcal{X} \to \mathcal{X}$ by

$$T_\lambda(x) = T(\lambda)(x) = \bar{T}(\lambda, x)$$

Theorem 2.3. *If $T : \Lambda \to \mathcal{M}(\mathcal{X}, \mathcal{X})$ is a uniform family of contractions on $\mathcal{X}$, then each map T_λ has a unique fixed point x_λ which depends continuously on λ. That is, the map $\lambda \to x_\lambda$ is a continuous map from Λ into $\mathcal{X}$.*

Proof. Let $g(\lambda)$ be the fixed point of the map T_λ which exists since the map T_λ is a contraction.

For $\lambda_1, \lambda_2 \in \Lambda$, we have

$$\begin{aligned} d(g(\lambda_1), g(\lambda_2)) =& d(T_{\lambda_1} g(\lambda_1), T_{\lambda_2} g(\lambda_2)) \\ \leq& d(T_{\lambda_1} g(\lambda_1), T_{\lambda_1} g(\lambda_2)) + d(T_{\lambda_1} g(\lambda_2), T_{\lambda_2} g(\lambda_2)) \\ \leq& \alpha d(g(\lambda_1), g(\lambda_2)) + d(T_{\lambda_1} g(\lambda_2), T_{\lambda_2} g(\lambda_2)). \end{aligned}$$

This implies that

$$d(g(\lambda_1), g(\lambda_2)) \leq (1-\alpha)^{-1} d(T_{\lambda_1} g(\lambda_2), T_{\lambda_2} g(\lambda_2)).$$

Since the map $\lambda \to T_\lambda g(\lambda_2)$ is continuous for fixed λ_2, we see that $\lambda \to g(\lambda)$ is continuous. □

There is another useful criterion for the existence of fixed points of transformations in Banach spaces.

Let X be a Banach space. Let $x, y \in X$. The line segment in X from x to y is the set of points $\{(1-t)x + ty : 0 \leq t \leq 1\}$. A subset F of X is called *convex* if for any two points $x, y \in F$, each point in the line segment from x to y is contained in F.

Examples.

(1) Linear subspaces are convex.
(2) Open and closed balls are convex.

The following are three remarkable theorems.

Theorem 2.4 (Brouwer Fixed Point Theorem)**.** *Every continuous map T of the closed unit ball in $\mathbb{R}^n$ to itself has a fixed point.*

Theorem 2.5 (Schauder Fixed Point Theorem)**.** *Every continuous self-map of a compact convex subset of a Banach space has a fixed point.*

Theorem 2.6 (Schauder-Tychonov Fixed Point Theorem)**.** *Every continuous self-map of a compact convex subset of a locally convex linear topological space to itself has a fixed point.*

Theorem 2.8 below is a generalization of the Schauder Fixed Point Theorem.

Definition 2.4. *Let E be a subset of a Banach space X. The* closed convex hull *of E, $\overline{\text{co}}(E)$, is the intersection of all closed convex sets which contain E.*

Clearly, $\overline{\text{co}}(E)$ is the smallest closed convex set containing E.

Theorem 2.7 (Mazur)**.** *The closed convex hull of a compact subset E of a Banach space is itself compact.*

Theorem 2.8 (Extended Schauder Fixed Point Theorem)**.** *Suppose $\mathcal{A}$ is a closed bounded convex subset of a Banach space and $T : \mathcal{A} \to \mathcal{A}$ is a continuous map such that the image $T\mathcal{A}$ of $\mathcal{A}$ has compact closure. Then, T has a fixed point in $\mathcal{A}$.*

Proof. Let $\mathcal{B} = T\mathcal{A}$. The closure of $\mathcal{B}$ is compact, so, by Mazur's theorem, $\overline{\text{co}}(\mathcal{B}) = \overline{\text{co}}(\text{closure}(\mathcal{B})) =: \mathcal{B}_1$ is also compact.

Since $\mathcal{B} \subseteq \mathcal{A}$, we have closure$(\mathcal{B}) \subset \mathcal{A}$ since $\mathcal{A}$ is closed, and $\mathcal{B}_1 \subseteq \mathcal{A}$ since $\mathcal{A}$ is convex. Thus, $T\mathcal{B}_1 \subseteq T\mathcal{A} = \mathcal{B} \subseteq \mathcal{B}_1$, so we may apply the Schauder Fixed Point Theorem to T on $\mathcal{B}_1$ to conclude that T has a fixed point in $\mathcal{B}_1$ which is, of course, also in $\mathcal{A}$. □