16. Hamiltonian Systems in $\mathbb{R}^{2n}$

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a C^k function, $k \geq 1$. Write coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ on $\mathbb{R}^{2n}$.

A system of differential equations of the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \ldots, n \tag{16.1}$$

is called a *Hamiltonian system* with n degrees of freedom and *Hamiltonian function* H. We also write X_H for the vector field defined by (16.1).

Sometimes we write the shortened form of (16.1) as

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q},$$

where $q = (q_1, \ldots, q_n), p = (p_1, \ldots, p_n)$. If we define $z = (q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$,

$$\nabla H = \Big(\frac{\partial H}{\partial q_1}, \ldots, \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n}\Big)$$

and

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where I is the $n \times n$ identity matrix, then (16.1) has the form

$$\dot{z} = J \nabla H(z). \tag{16.2}$$

The matrix J above is called the standard symplectic matrix. It is one of the normal forms of a non-degenerate alternating bilinear form on $\mathbb{R}^{2n}$. Because of equation (16.2), one sometimes refers to a Hamiltonian system as a symplectic gradient. However, the orbit structure of a Hamiltonian system is vastly different from that of a gradient system.

Proposition 16.1. *If X_H is a Hamiltonian system with Hamiltonian H, then H is constant on orbits.*

Proof. For any solution curve $\gamma(t) = (q(t), p(t))$ we have

$$\begin{aligned} \frac{dH(q(t), p(t))}{dt} &= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i \\ &= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}\Big(-\frac{\partial H}{\partial q_i}\Big) = 0. \end{aligned}$$

□

Definition 16.1. *A C^1 (nonconstant) function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a* first integral *for a differential equation $\dot{x} = f(x)$ defined on $\mathbb{R}^n$ if Φ is constant on any orbits.*

The above proposition says that the Hamiltonian function is a first integral of the corresponding Hamiltonian system.

Definition 16.2. *A map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is* volume preserving *if for any measurable set $E \subset \mathbb{R}^n$, $\phi^{-1}(E)$ and E have the same Lebesgue measure.*

Proposition 16.2. *A C^1 diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is volume preserving if and only if the Jacobian is ± 1 everywhere.*

Proof. Recall the formula of change of variable

$$\int_E f(y)dy = \int_{\phi^{-1}(E)} f(\phi(x))\Big|\det \frac{\partial y}{\partial x}\Big|dx,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. In particular, if we take $f = 1$, then we have

$$\int_E dy = \int_{\phi^{-1}(E)} \Big|\det \frac{\partial y}{\partial x}\Big|dx,$$

So if $\det \dfrac{\partial y}{\partial x} = \pm 1$, then we have $m(E) = m(\phi^{-1}(E))$ for any measurable set E, where m denotes the Lebesgue measure. On the other hand, for example, if $\det \dfrac{\partial y}{\partial x}(x) > 1$ at some point x, then by continuity we have that $\det \dfrac{\partial y}{\partial x} > 1$ in a neighborhood U of x. Hence we get that

$$m(\phi(U)) = \int_{\phi(U)} dy = \int_U \Big|\det \frac{\partial y}{\partial x}\Big|dx > \int_U dx = m(U),$$

that is, ϕ cannot be measure preserving. □

Proposition 16.3. *Suppose $\phi(t, x)$ is a solution of the initial value problem*

$$\dot{x} = f(x), \qquad x(0) = x,$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field. Then $\phi(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is volume preserving if and only if $\operatorname{div} f = 0$ *everywhere.*

Proof. Recall that $\dfrac{\partial}{\partial x}\phi(t, x)$ satisfies the initial value problem

$$\dot{Z} = \frac{\partial}{\partial x} f(\phi(t, x)) \cdot Z, \qquad \frac{\partial}{\partial x}\phi(t, x)|_{t=t_0} = \mathrm{id}\,.$$

This is a linear equation with respect to $Z(t)$ and of $\dfrac{\partial}{\partial x}\phi(t, x)$ is in fact a fundamental matrix, and the Wronskian is $W(t) = W(t, x) =$

$\det \dfrac{\partial}{\partial x}\phi(t,x)$, which is the Jacobian of $\phi(t,\cdot)$. Note that for any $t_0, t \in \mathbb{R}$,

$$W(t) = W(t_0) \exp \int_{t_0}^{t} \operatorname{tr} \frac{\partial}{\partial x} f(\phi(s,x))ds,$$

and $\operatorname{tr} \dfrac{\partial}{\partial x} f(x) = \operatorname{div} f(x)$. So we get that for any t, $\phi(t,\cdot)$ is measure preserving if and only if $W(t,x) = 1$ and if and only if $\operatorname{div} f(x) = 0$ everywhere. □

Proposition 16.4. *If X_H is a Hamiltonian system with Hamiltonian H, and $\phi(t,x)$ be the solution satisfying $\phi(0,x) = x$. Then the map $\phi(t,\cdot) : \mathbb{R}^{2n} \to R^{2n}$ is volume preserving.*

Proof. Since

$$X_H = \Big(\frac{\partial H}{\partial p_1}, \dots \frac{\partial H}{\partial p_n}, \frac{\partial H}{\partial q_1}, \dots - \frac{\partial H}{\partial q_n}\Big)$$

We have

$$\operatorname{tr} DX_H = \sum_{i=1}^{n} \Big(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i},\Big) = 0$$

That is, $\operatorname{div} X_H = 0$. □

Classical Mechanical Systems in $\mathbb{R}^n$.

Let $x = (x_1, \dots, x_n)$ denote points in $\mathbb{R}^n$, and let $U : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Let $m_i > 0, i = 1, \dots, n$ be n positive real constants.

The system

$$m_i \ddot{x}_i = -\frac{\partial U}{\partial x_i}, \quad i = 1, \dots, n \tag{16.3}$$

is called a *conservative mechanical system* with *potential function U* in $\mathbb{R}^n$. The constants represent the masses of the system, and the function U plays the role of potential energy. The system (16.3) is a formulation of Newton's law of motion which, in words, says that *mass times acceleration* equals *force* and the force is the negative gradient of the potential energy function. Note that the potential function is a function of position alone (not velocity) and can be an arbitrary C^1 function.

Let $c_i > 0, i = 1, \dots, n$, denote some other constants.

The system

$$m_i \ddot{x}_i + c_i \dot{x}_i = -\frac{\partial U}{\partial x_i}, \qquad i = 1, \dots, n, \tag{16.4}$$

is called a *dissipative mechanical system* with potential function U and frictional constants c_i.

Given (16.3) or (16.4), we set $v = (v_1, \dots, v_n) = (\dot{x}_1, \dots, \dot{x}_n)$ and form the function

$$T(x, v) = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 + U(x).$$

This is called the *total energy function*, or simply the *energy function*, of the system. The function $K(v) = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2$ is called the *Kinetic Energy* of the system. It is a function of velocity alone.

The equations (16.3), (16.4) are second order systems. We can form the associated first order systems

$$\begin{aligned} \dot{x}_i &= \quad v_i, \\ m_i \dot{v}_i &= -\frac{\partial U}{\partial x_i}; \end{aligned} \tag{16.5}$$

and

$$\begin{aligned} \dot{x}_i &= v_i, \\ m_i \dot{v}_i &= -c_i v_i - \frac{\partial U}{\partial x_i}. \end{aligned} \tag{16.6}$$

Proposition 16.5. *There is a coordinate system on $\mathbb{R}^{2n}$ in which the conservative system (16.3) becomes a Hamiltonian system.*

Proof. Let $q_i = x_i, p_i = m_i v_i$. Then,

$$H(q, p) = T(x, v) = \frac{1}{2} \sum_{i=1}^{n} \frac{p_i^2}{m_i} + U(q_1, \dots, q_n)$$

and (16.3) becomes

$$\begin{aligned} \dot{q}_i &= \quad \frac{p_i}{m_i} = \quad \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= m_i \dot{v}_i = -\frac{\partial H}{\partial q_i}. \end{aligned}$$

□

Fact 16.6. *(1) The critical points of a classical mechanical system are the points (x, v) with x a critical point of U and $v = 0$.*

(2) The total energy function $T(x, v)$ is a Lyapunov function for a conservative mechanical system and a strict Lyapunov function for a dissipative mechanical system.

(3) If x_0 is a strict relative minimum of the potential function U, then $(x_0, 0)$ is a stable equilibrium of the system (16.5) and an asymptotically stable equilibrium of the system (16.6).

The fact that the energy function $T(x, v)$ is a Lyapunov function for a mechanical system frequently helps us to get a picture of the the solutions without solving the equation.

To illustrate this phenomenon, let us consider systems with one degree of freedom. These have the form

$$\ddot{x} + f(x) = 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a real-valued function of one real variable.

Writing $U(x) = \int_0^x f(s)ds$, we get a total energy function of the form

$$T(x, v) = \frac{1}{2}v^2 + U(x).$$

Let us consider some examples.

Examples.

1. (Harmonic oscillator)

$$T(x, v) = \frac{v^2}{2} + \frac{x^2}{2}.$$

The orbits are circles around the origin $(0, 0)$ which is a single stable equilibrium.

2. (Pendulum)

$$T(x, v) = \frac{v^2}{2} + k(1 - \cos(x))$$

for some constant $k > 0$.

The critical points are $(\pm n\pi, 0)$. The stable ones are $(2\pi n, 0)$ and the saddles are $(\pi(2n + 1), 0)$.

3. (Duffing equation)

$$T(x, v) = \frac{v^2}{2} + \frac{x^4}{4} - \frac{x^2}{2}.$$

There are three critical points at $(0, 0), (-1, 0), (1, 0)$. The origin is a saddle and the others are centers

When one adds friction to each of the above equations, the orbits cross the level sets of T instead of lying in them.

Hamiltonian Systems and Variational Problems.

We have seen that Hamiltonian systems arise naturally in Classical Mechanics. Now we will see that they also arise in general problems in the Calculus of Variations.

Consider a real-valued function $L(q, \dot{q}, t)$ of the variables $(q, \dot{q}, t) \in \mathbb{R}^{2n+1}$. Let $t_1 < t_2$ be real numbers, a, b be two fixed elements in $\mathbb{R}^n$, and suppose we seek to find conditions on C^2 curves $\gamma : q = q(t)$ defined on the interval $[t_1, t_2]$ such that

$$q(t_1) = a, \quad q(t_2) = b$$

and

$$I(\gamma) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)dt \tag{16.7}$$

is stationary for nearby curves η with the same boundary conditions (**??**) and $\dot{q}(t)$ is the derivative $\frac{dq}{dt}(t)$.

This means that we consider one-parameter families $q(t, \alpha)$ of C^2 curves with $q(t, 0) = \gamma(t)$ such that

$$q(t_1, \alpha) = a, \quad q(t_2, \alpha) = b \tag{16.8}$$

for all α and

$$\frac{dI}{d\alpha}\Big|_{\alpha=0} = 0 \tag{16.9}$$

for

$$I(\alpha) = \int_{t_1}^{t_2} L(q(t, \alpha), \dot{q}(t, \alpha), t)dt.$$

One sometimes writes the condition (16.9) as

$$\delta \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)dt = 0.$$

The operator δ is used to denote the fact that we are not considering an ordinary derivative, but rather, a stationary value of the integral as a family of curves changes.

Note that if γ were a curve for which the integral (16.7) assumed a minimum for all nearby curves with the given boundary conditions, then it would be stationary in the sense of condition (16.9).

To express the derivative $\dfrac{dI}{d\alpha}$ more conveniently, we introduce some notation. Write $q = (q_1, \ldots, q_n), \dot{q} = (\dot{q}_1, \ldots, \dot{q}_n)$, L_{q_k} for the partial derivative of L with respect to q_k, $L_{\dot{q}_k}$ for the partial derivative of L with respect to $\dot{q}_k$ with $1 \leq k \leq n$. Also, we denote differentiation with respect to t by "dot" and that with respect to α by "prime".

Consider the condition $\frac{dI}{d\alpha}\Big|_{\alpha=0} = 0$. We have

$$0 = I'(\alpha) = \int_{t_1}^{t_2} L_q \cdot q' + L_{\dot{q}} \cdot \dot{q}' dt \tag{16.10}$$

where $L_q \cdot q'$, $L_{\dot{q}} \cdot \dot{q}'$ respectively stand for

$$\sum_{k=1}^{n} L_{q_k} q_k' \qquad \text{and} \qquad \sum_{k=1}^{n} L_{\dot{q}_k} \dot{q}_k'.$$

By (16.8) we have $q'(t_1, \alpha) = q'(t_1, \alpha) = 0$. Hence $L_{\dot{q}} \cdot q'|_{t_1}^{t_2} = 0$. Integrating by parts we have

$$\int_{t_1}^{t_2} L_{\dot{q}} \cdot \dot{q}' dt = L_{\dot{q}} \cdot q'|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{dL_{\dot{q}}}{dt} \cdot q' dt = -\int_{t_1}^{t_2} \frac{dL_{\dot{q}}}{dt} \cdot q' dt.$$

Hence, (16.10) becomes

$$0 = \frac{dI}{d\alpha}\Big|_{\alpha=0} = \int_{t_1}^{t_2} (L_q - \frac{d}{dt} L_{\dot{q}}) q' dt.$$

Now, the "prime" derivatives q' can be made arbitrary, so the equation implies

$$L_q - \frac{d}{dt} L_{\dot{q}} = 0, \tag{16.11}$$

or, written out completely,

$$\frac{d}{dt} L_{\dot{q}_k} = L_{q_k}, \quad k = 1, \ldots, n. \tag{16.12}$$

The equations (16.12) are called the *Euler-Lagrange equations*.

At a curve $(q(t), \dot{q}(t), t)$ which makes the integral (16.7) stationary, we have that $(q(t), \dot{q}(t))$ satisfies

$$\frac{d}{dt} L_{\dot{q}_k}(q(t), \dot{q}(t), t) = L_{q_k}(q(t), \dot{q}(t), t), \quad \forall k = 1, \ldots, n.$$

Note that these are second order differential equations.

Lemma 16.7. *The Euler-Lagrange equations have a first integral*

$$L - \dot{q} L_{\dot{q}}.$$

Proof. This is because

$$\begin{aligned} \frac{d}{dt}(L - \dot{q} L_{\dot{q}}) =& L_q \dot{q} + L_{\dot{q}} \ddot{q} - \ddot{q} L_{\dot{q}} - \dot{q} \frac{dL_{\dot{q}}}{dt} \\ =& \dot{q} \Big(L_q - \frac{dL_{\dot{q}}}{dt} \Big). \end{aligned}$$

□

We now show that if in an open set G in $\mathbb{R}^{2n+1}$ the matrix function

$$L_{\dot{q}_k,\dot{q}_\ell}(q,\dot{q},t) \tag{16.13}$$

is non-singular, then we can choose coordinates in which the Euler-Lagrange equations become a Hamiltonian system. This is one of the main reasons that Hamiltonian systems are important.

So, assume that we have the independent coordinates $(q,\dot{q},t)$ in an open set G in $\mathbb{R}^{2n+1}$, that $L(q,\dot{q},t)$ is a C^2 real-valued function in G, and that the matrix function (16.13) is non-singular in G.

Consider the set of equations

$$p_k = L_{\dot{q}_k}(q,\dot{q},t), \qquad k = 1,\ldots,n. \tag{16.14}$$

Because of the assumption that (16.13) is non-singular, the Implicit Function Theorem gives us a set of C^2 functions $S_k(q,p,t)$ for $k = 1,\ldots,n$, such that (16.14) holds if and only if

$$\dot{q}_k = S_k(q,p,t), \qquad k = 1,\ldots,n. \tag{16.15}$$

Let

$$\begin{aligned} H(q,p,t) &= \sum_k p_k\dot{q}_k - L(q,\dot{q},t) \\ &= \sum_k p_k S_k(q,p,t) - L(q,S(q,p,t),t). \end{aligned}$$

Then, for $\ell = 1,\ldots,n$,

$$\begin{aligned} \frac{\partial H}{\partial p_\ell} &= S_\ell(q,p,t) + \sum_k p_k \frac{\partial S_k(q,p,t)}{\partial p_\ell} - \sum_k L_{\dot{q}_k}\frac{\partial S_k}{\partial p_\ell} \\ &= S_\ell(q,p,t) = \dot{q}_\ell \end{aligned}$$

since $p_k = L_{\dot{q}_k}(q,\dot{q},t)$. Also,

$$\begin{aligned} -\frac{\partial H}{\partial q_\ell} &= -\sum_k p_k \frac{\partial S_k(q,p,t)}{\partial q_\ell} + L_{q_\ell} + \sum_k L_{\dot{q}_k}\frac{\partial S_k}{\partial q_\ell} \\ &= L_{q_\ell} \\ &= \frac{d}{dt}L_{\dot{q}_\ell} \qquad \text{(by Euler-Lagrange)} \\ &= \frac{d}{dt}p_\ell \qquad \text{(by definition of } p_\ell). \end{aligned}$$

In the (q,p,t), coordinates, we therefore have a Hamiltonian system with Hamiltonian function H. If $L(q,\dot{q},t) = L(q,\dot{q})$ is independent of time t, then so is H. However, in the general case, both L and H are time dependent. Note that if H is time-dependent, then the function H is *not* constant on solutions to the Hamiltonian system.

Let us now return to the Conservative Mechanical system with potential energy U we studied previously.

Using, position q and momentum p as coordinates, we saw that the equations of motion were a (time-independent) Hamiltonian system with Hamiltonian function

$$H(q,p) = \frac{1}{2}\sum_k \frac{p_k^2}{m_k} + U(q_1,\ldots,q_n)$$

and the velocity $\dot{q}$ satisfied $\dot{q}_k = \dfrac{p_k}{m_k}$.

If we assume that this Hamiltonian system comes from a variational problem as above, we are led to write

$$H = \sum_k p_k\dot{q}_k - L(q,p),$$

or

$$\begin{aligned} L &= \sum_k p_k\dot{q}_k - H = \sum_k p_k\dot{q}_k - \frac{1}{2}\sum_k \frac{p_k^2}{m_k} - U \\ &= \sum_k p_k\frac{p_k}{m_k} - \frac{1}{2}\sum_k \frac{p_k^2}{m_k} - U = \frac{1}{2}\sum_k \frac{p_k^2}{m_k} - U = K - U, \end{aligned}$$

where K denotes the kinetic energy. The function $L = K - U$ is called the *Lagrangian function* or *action function*, as opposed to the function $T = K + U$, which was called the *Energy function*.

From the above, we are led to guess that conservative mechanical systems would satisfy the Euler-Lagrange equations for the function $L = K - U$. This is indeed the case as can be easily verified. In this case, it can be verified that the integral

$$\int L(q,\dot{q})dt$$

is actually minimized by the solution curves $(q,\dot{q})$, not just made stationary. This is known as Hamilton's *Principal of Least Action*.

Examples.

1. (Shortest curve) Suppose that we consider the problem of finding the curve γ of shortest length shortest joining two points $a, b \in \mathbb{R}^2$.

Writing $\gamma(t) = (x(t), y(t))$, $t_1 \le t \le t_2$, we seek to minimize the function

$$I(\gamma) = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}dt$$

over all such curves.

Let $L(x, y, \dot{x}, \dot{y}, t) = \sqrt{\dot{x}^2 + \dot{y}^2}$. The Euler-Lagrange equations become

$$\frac{d}{dt}L_{\dot{x}} = L_x, \qquad \frac{d}{dt}L_{\dot{y}} = L_y.$$

Since, L is independent of x, y, t, we get

$$\frac{d}{dt}L_{\dot{x}} = 0, \ \frac{d}{dt}L_{\dot{y}} = 0, \ L_t = 0.$$

These equations become

$$\frac{d}{dt}L_{\dot{x}} = \frac{d}{dt}\frac{\dot{x}}{L} = \frac{L\ddot{x} - \dot{x}L_t}{L^2} = \frac{\ddot{x}}{L} = 0,$$

and

$$\frac{d}{dt}L_{\dot{y}} = \frac{d}{dt}\frac{\dot{y}}{L} = \frac{L\ddot{y} - \dot{y}L_t}{L^2} = \frac{\ddot{y}}{L} = 0.$$

Using that L is never zero, we see that the only solutions are those $(x(t), y(t))$ for which $\ddot{x} = 0$, $\ddot{y} = 0$. That is, the only solutions are straight lines. With the above boundary condition, we get a unique line segment joining a to b.

2. (Minimal surface of revolution) We consider the problem of finding the curve γ joint two points $a, b \in \mathbb{R}^2$ that has minimal surface of revolution about the x-axis.

Writing $\gamma(t) = (x(t), y(t))$, $t_1 \le t \le t_2$, we seek to minimize the function

$$S(\gamma) = 2\pi \int_{t_1}^{t_2} y\sqrt{1 + \dot{y}^2}dx$$

over all such curves.

Replacing t and q by x and y respectively, we get

$$L(x, y, \dot{y}) = L(y, \dot{y}) = y\sqrt{1 + \dot{y}^2}.$$

The Euler-Lagrange equation is

$$L_y = \frac{dL_{\dot{y}}}{dx},$$

and by Lemma 16.7 it has a first integral

$$y\sqrt{1 + \dot{y}^2} - \dot{y} \cdot y\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = c_1,$$

or

$$\frac{y}{\sqrt{1 + \dot{y}^2}} = c_1$$

for some constant c_1. Let $\dot{y} = \sinh\tau$, then the equation gives $y = c_1\cosh\tau$. Hence

$$dx = \frac{dy}{\dot{y}} = \frac{c_1\sinh\tau d\tau}{\sinh\tau} = c_1 d\tau,$$

and we get $x = c_1\tau + c_2$. So

$$y = c_1\cosh\frac{x-c_2}{c_1}.$$

The is chain curve, and the constants c_1 and c_2 can be determined by using the fact that the curve passing through points $a, b \in \mathbb{R}^2$.

This curve is also known as a *chain curve* or *catenary*.

3. (Brachistochrone curve, or curve of fastest descent) We consider the problem of finding the curve γ joint two points $a, b \in \mathbb{R}^2$ such that a particle will slide from a to b in the least amount of time.

Take coordinate system such that a is at the origin, and the y-axis is oriented downwards, and the x-axis is oriented in the direction that makes the point b in the first quadrant.

For a curve joint points a and b, we take the arc length s as a parameter. Since the kinetic energy of the particle is equal to the change of the potential energy, $mgy = \frac{1}{2}mv^2$, that is, $v = \sqrt{2gy}$. So we get

$$\frac{ds}{dt} = v = \sqrt{2gy},$$

or

$$dt = \frac{ds}{v} = \frac{\sqrt{1+\dot{y}^2}dx}{\sqrt{2gy}}.$$

Hence, the time needed for the particle travelling from point a to point b along a curve γ is given by

$$T(\gamma) = \frac{1}{\sqrt{2g}}\int_0^{x_b}\frac{\sqrt{1+\dot{y}^2}}{\sqrt{y}}dx,$$

where x_b is the x-coordinate of b. We will minimize the function over curves from a to b.

Replacing t and q by x and y respectively, we get

$$L(x, y, \dot{y}) = L(y, \dot{y}) = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{y}}.$$

By Lemma 16.7 a first integral of the Euler-Lagrange equation is

$$\frac{\sqrt{1+\dot{y}^2}}{\sqrt{y}} - \dot{y}\cdot\frac{\dot{y}}{\sqrt{y}\sqrt{1+\dot{y}^2}} = \sqrt{c},$$

or

$$\frac{1}{\sqrt{y}\sqrt{1+\dot{y}^2}} = \sqrt{c}$$

for some constant $c > 0$. We can write the equation as

$$c = y(1+\dot{y}^2).$$

Let $\dot{y} = \cot\tau$, then

$$y = \frac{c}{1+\dot{y}^2} = \frac{c}{1+\cot^2\tau} = c\sin^2\tau = \frac{c}{2}(1-\cos 2\tau).$$

Also, since

$$dx = \frac{dy}{\dot{y}} = \frac{2c\sin\tau\cos\tau d\tau}{\cot\tau} = 2c\sin^2\tau d\tau = c(1-\cos 2\tau)d\tau,$$

we have

$$x = c\Big(\tau - \frac{1}{2}\sin 2\tau\Big) + c' = \frac{c}{2}(2\tau - \sin 2\tau) + c'.$$

Since we assume that a is at the origin, $x = 0$ at $\tau = 0$. So $c' = 0$. Replace 2τ by θ and c by $2A$, we have

$$\begin{aligned} x =& A(\theta - \sin\theta) \\ y =& A(1-\cos\theta). \end{aligned} \tag{16.16}$$

This curve is also known as a *cycloid*.