3. General Properties of Differential Equations

Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space and let (t, x) denote coordinates in $\mathbb{R}^{n+1}$ with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Write $\dot{x} = \dfrac{dx}{dt}$.

A first order ordinary differential equation in $\mathbb{R}^n$ is an expression of the form

$$\dot{x} = f(t, x), \tag{3.1}$$

where f is a function from an open set $D \subseteq \mathbb{R}^{n+1}$ to $\mathbb{R}^n$. When f depends explicitly on t, the equation (3.1) is called *nonautonomous* or *time dependent*. If f is independent of t, it is called *autonomous* or *time independent*.

A *solution* to (3.1) is a differentiable function $x(t)$ from a real interval I into $\mathbb{R}^n$ so that

(1) $\{(t, x(t)) : t \in I\} \subseteq D$,
(2) For $t \in I, \dot{x}(t) = f(t, x(t))$.

If we fix a point $(t_0, x_0) \in D$, we are sometimes interested in solutions $x(\cdot)$ of (3.1) for which $x(t_0) = x_0$.

This leads us to the system of equations

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \tag{3.2}$$

which we will call *the initial value problem* of the differential equation (3.1) with initial value (t_0, x_0), or simply the initial value problem.

Remarks.

(1) The $n-$th order scalar differential equation

$$\frac{d^n x}{dt^n} = g(t, x, \dot{x}, \frac{d^2 x}{dt^2}, \ldots, \frac{d^{n-1} x}{dt^{n-1}})$$

can be written as the vector system

$$\begin{aligned} \dot{x} &= x_1 \\ \frac{dx_1}{dt} &= x_2 \\ &\vdots \\ \frac{dx_{n-1}}{dt} &= g(t, x, x_1, \ldots, x_{n-1}) \end{aligned}$$

using the vector, $(t, x, x_1, \ldots, x_{n-1})$ with $x_i = \dfrac{d^i x}{dt^i}$ so it is usually not necessary to consider higher order differential equations for general properties.

(2) In issues in which $f(t,x)$ is very smooth, e.g. C^∞, it is frequently useful to replace the non-autonomous equation (3.1) by the system $\dot{t} = 1, \dot{x} = f(t,x)$ and obtain an autonomous equation in one higher dimension.

Examples.

(1) The first example shows that even if the right hand side of a differential equation is a polynomial, solutions to (3.1) may not be defined for all real time.

Let $D = \mathbb{R}^2, f(t,x) = x^2$. The initial value problem

$$\dot{x} = x^2, \quad x(0) = x_0$$

has the unique solution $\phi(t) = -\dfrac{1}{t - x_0^{-1}}$ for $x_0 \neq 0$ and $\phi(t) = 0\ \forall t$ for $x_0 = 0$. For $x_0 \neq 0$, these solutions blow up in finite time.

(2) The second example shows that the initial value problem of a continuous differential equation need not have a unique solution.

Let $D = \mathbb{R}^2$,

$$f(t,x) = \begin{cases} \sqrt{x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Fix a real number $c > 0$, and define the function

$$\phi_c(t) = \begin{cases} \frac{(t-c)^2}{4} & \text{for } t \geq c, \\ 0 & \text{for } t < c. \end{cases}$$

Then, each $\phi_c(t)$ is a solution to $\dot{x} = f(t,x)$ with value 0 at $t_0 = 0$.

Lemma 3.1. *Suppose that $f(t,x)$ is a continuous function on an open set D in $\mathbb{R}^{n+1}$. Let $(t_0, x_0) \in D$. Then, a continuous function $x(t)$ is a solution to the single integral equation*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{3.3}$$

if and only if it is a solution to the initial value problem (3.2).

Proof. "$\Longrightarrow$" Suppose that $x(\cdot)$ is a continuous function which solves the integral equation. Then, $x(t_0) = x_0$, and since f is continuous, the Fundamental Theorem of Calculus gives that $x(t)$ is differentiable with

$$\dot{x} = f(t, x(t))$$

so that $x(\cdot)$ solves (3.2).

"$\Longleftarrow$" Conversely, suppose that the $x(\cdot)$ is a solution to the problem (3.2). Then, $x(\cdot)$ is differentiable, hence continuous, on an interval about t_0. Let $h(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds$.

Again the Fundamental Theorem of Calculus gives that h is differentiable with derivative $f(t, x(t))$ at t. Thus, both $x(t)$ and $h(t)$ are differentiable functions with the same derivative on an interval about t_0. Hence, they differ by a constant. But they both have the value x_0 at t_0, so the constant is 0, and $x(t)$ solves the integral equation. $\square$

We wish to show that differential equations with continuous right hand sides have solutions at least on small intervals.

Theorem 3.2 (Peano Existence Theorem). *Suppose that $f(t, x)$ is continuous in the open set $D \subseteq \mathbb{R}^1 \times \mathbb{R}^n$. Then, for (t_0, x_0) in D, the initial value problem (3.2) has at least one solution.*

We will give two proofs of this theorem. The first depends on a theorem in Functional Analysis.

Proof 1 of Peano Theorem. For $\alpha > 0, \beta > 0$ let

$$\begin{aligned} I_\alpha &= I_\alpha(t_0) = \{t : |t - t_0| \leq \alpha\}, \\ B_\beta &= B_\beta(x_0) = \{x : |x - x_0| \leq \beta\}. \end{aligned}$$

Choose α, β small enough so that $I_\alpha \times B_\beta \subseteq D$.

Since $I_\alpha \times B_\beta$ is compact and f is continuous on $I_\alpha \times B_\beta$, the quantity

$$M = \sup\{|f(t, x)| : (t, x) \in I_\alpha \times B_\beta\}$$

is finite.

Let α_1 be positive and small enough so that $M\alpha_1 \leq \beta$.

Let

$$\mathcal{A} = \{\phi \in \mathcal{C}(I_{\alpha_1}, \mathbb{R}^n) : \phi(t_0) = x_0, |\phi(t) - x_0| \leq \beta \ \forall t \in I_{\alpha_1}\}.$$

Clearly $\mathcal{A}$ is a closed bounded convex subset of the Banach space $\mathcal{C}(I_{\alpha_1}, \mathbb{R}^n)$ with the sup norm. Let $T : \mathcal{C}(I_{\alpha_1}, \mathbb{R}^n) \to \mathcal{C}(I_{\alpha_1}, \mathbb{R}^n)$ be defined by

$$(T\phi)(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds.$$

By the claims below, the Extended Schauder Fixed Point Theorem gives us a fixed point ψ of T in $\mathcal{A}$. This fixed point solves the integral equation (3.3), so it provides a solution to the IVP (3.2). $\square$

Claim 1. *T maps $\mathcal{A}$ into itself.*

Claim 2. *T is continuous*

Claim 3. *$T\mathcal{A}$ has compact closure.*

Proof of Claim 1. Let $\phi \in \mathcal{A}$. Clearly, $I_{\alpha_1} \times \phi(I_{\alpha_1}) \subseteq D$ so T is well-defined. Also, $(T\phi)(t_0) = x_0$. Next, for $t \in I_{\alpha_1}$,

$$\left|(T\phi)(t) - x_0\right| \leq \left|\int_{t_0}^{t} f(s, \phi(s))ds\right| \leq M\alpha_1 \leq \beta.$$

Hence, $T\phi \in \mathcal{A}$. □

Proof of Claim 2. Let $\epsilon > 0$. We know that f is uniformly continuous on $I_{\alpha_1} \times B_\beta$. Take $\delta > 0$ such that if $|(t,x)-(s,y)| < \delta$ and $(t,x),(s,y) \in I_{\alpha_1} \times B_\beta$, then, $|f(t,x) - f(s,y)| < \epsilon/\alpha_1$.

Suppose that $\phi, \psi \in \mathcal{A}$ are such that $\|\phi - \psi\| < \delta$. This means that, for each $t \in I_{\alpha_1}$, $|\phi(t) - \psi(t)| < \delta$. Thus, for $t \in I_{\alpha_1}$,

$$\begin{aligned}\left|(T\phi)(t) - (T\psi)(t)\right| &\leq \left|\int_{t_0}^{t} f(s,\phi(s)) - f(s,\psi(s))ds\right| \\ &< (\epsilon/\alpha_1)|t - t_0| \leq (\epsilon/\alpha_1) \cdot \alpha_1 = \epsilon.\end{aligned}$$

Hence, $\|T\phi - T\psi\| \leq \epsilon$. So T is continuous on $\mathcal{A}$. □

Proof of Claim 3. First, note that $T\mathcal{A}$ is equicontinuous. In fact, for any $\phi \in \mathcal{A}$, $t, u \in I_{\alpha_1}$,

$$\left|(T\phi)(t) - (T\phi)(u)\right| \leq \left|\int_{u}^{t} f(s,\phi(s))ds\right| \leq M|t-u|.$$

It follows that the closure of $T\mathcal{A}$ is also equicontinuous. Since it is also bounded, it will follow from the Arzela-Ascoli Theorem that $T\mathcal{A}$ has compact closure as required. □

Proof 2 of Peano Theorem. Let $I_\alpha, I_{\alpha_1}, B_\beta$ be as in Proof 1.

Take $n \geq 1$, and let $h = h_n = \dfrac{\alpha_1}{n}$.

We will consider the Euler polygonal approximations ϕ_h for solutions defined in the following way.

First, let $t_1 = t_0 + h$ and $x_1 = x_0 + f(t_0, x_0)h$. Then for $1 \leq i \leq n-1$, let

$$t_{i+1} = t_i + h = t_0 + ih, \qquad x_{i+1} = x_i + f(t_i, x_i)h. \tag{3.4}$$

This is a discrete sequence of vectors. Interpolate linearly between (t_i, x_i) and (t_{i+1}, x_{i+1}) to form the function

$$\phi_h(t) = x_i + f(t_i, x_i)(t - t_i) \quad \text{for } t_i \leq t \leq t_{i+1}. \tag{3.5}$$

By Claim 1 below, all x_i are in B_β and hence $|f(t_i, x_i)| \leq M$. By definition, ϕ_h is a linear function on each interval $[t_i, t_{i+1}]$ for $i = 0, \ldots, n-1$ with slope $f(t_i, x_i)$. So ϕ_h is a Lipschitz function with Lipschitz constant less than or equal to M. Since this is true for any n, the sequence

$\{\phi_{h_n}\}$ is equicontinuous. Also, the fact that $x_i \in B_\beta$ and ϕ_{h_n} is piecewise linear for any n implies that $\{\phi_{h_n}\}$ are uniformly bounded.

Thus, by the Arzela-Ascoli theorem, there is a sequence $\phi_{h_{n_j}}$ which converges to a function ψ defined on I_{α_1}. Claim 2 below gives that

$$\left|\psi(t) - x_0 - \int_{t_0}^{t} f(s, \psi(s))ds\right| = 0,$$

or equivalently,

$$\psi(t) = x_0 + \int_{t_0}^{t} f(s, \psi(s))ds.$$

That is, ψ is a solution of (3.2). □

Claim 1. $|x_i - x_0| \leq \beta$ *for any* $i = 0, 1, \ldots, n$.

Claim 2. *As* $h_n \to 0$, $\left|\phi_{h_n}(t) - x_0 - \int_{t_0}^{t} f(s, \phi_{h_n}(s))ds\right| \to 0.$

Proof of Claim 1. We prove it by induction.

Clearly it is true for $i = 0$.

Suppose $|x_j - x_0| \leq \beta$ for all $j = 0, 1, \ldots, i$. Then $(t_j, x_j) \in I_{\alpha_1} \times B_\beta$ and therefore

$$|f(t_j, x_j)| \leq M.$$

So

$$\begin{aligned} |x_{i+1} - x_0| =& |x_i + f(t_i, x_i)h - x_0| \\ =& \left|x_0 + \sum_{j=0}^{i} f(t_j, x_j)h - x_0\right| \\ \leq& M \cdot (i+1)h \leq M \cdot \alpha_1 \leq \beta, \end{aligned}$$

whenever $i < n$. Now the claim follows from induction. □

Proof of Claim 2. We leave it as an exercise. □