4.1. Existence and Uniqueness Theorem. We will now see that rather mild conditions on the right hand side of an ordinary differential equation give us local existence and uniqueness of solutions.

Definition 4.1. *Let $f : D \to \mathbb{R}^n$ be a coninuous function defined in the open set $D \subseteq \mathbb{R} \times \mathbb{R}^n$. We say that f is* locally Lipschitz in the $\mathbb{R}^n$ variable *if for each $(t_0, x_0) \in D$, there is an open set $U \subseteq D$ containing (t_0, x_0) and a constant $K > 0$ such that if $(t, x), (t, y) \in U$, then*

$$|f(t, x) - f(t, y)| \leq K|x - y|.$$

If we write f as $f(t, x)$ with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we also say that f is locally Lipschitz in x.

Remark 4.1. *If $f(t, x)$ is C^1 in x, with derivative depending continuously on t, then it is locally Lipschitz in x.*

Theorem 4.2 (Existence and Uniqueness Theorem for ODE)**.** *Suppose $f(t, x)$ is continuous in an open set $D \subseteq \mathbb{R} \times \mathbb{R}^n$ and is locally Lipschitz in x in D. Let $(t_0, x_0) \in D$. Then the initial value problem*

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{4.1}$$

has a unique solution defined in a small interval I about t_0 in $\mathbb{R}$.

Proof. Let U be an open neighborhood about (t_0, x_0) in D so that

(i) f is continuous in U and Lipschitz in x in U with Lipschitz constant no larger than $K > 0$.
(ii) $|f(t, x)| \leq M$ for $(t, x) \in U$.

Let $I_\alpha = \{t : |t - t_0| \leq \alpha\}, B_\beta = \{x : |x - x_0| \leq \beta\}$. Choose α, β small enough so that $I_\alpha \times B_\beta \subseteq U$.

Let α_0 be small enough so that

$$\alpha_0 M < \beta \tag{4.2}$$

and

$$\alpha_0 K < 1 \tag{4.3}$$

Now, consider the set

$$\mathcal{A} = \{\phi \in \mathcal{C}(I_{\alpha_0}, \mathbb{R}^n) : \phi(t_0) = x_0, |\phi(t) - x_0| \leq \beta \ \forall t \in I_{\alpha_0}\}. \tag{4.4}$$

With the sup norm, $\mathcal{A}$ is a closed bounded subset of the Banach space $\mathcal{C}(I_{\alpha_0}, \mathbb{R}^n)$ of continuous functions from I_{α_0} into $\mathbb{R}^n$. Thus, $\mathcal{A}$ is a complete metric space with the metric $d(\phi, \psi) = \sup_{t \in I_{\alpha_0}} |\phi(t) - \psi(t)|$.

Consider again the integral operator $T : C(I_{\alpha_0}, \mathbb{R}^n) \to C(I_{\alpha_0}, \mathbb{R}^n)$ given by

$$(T\phi)(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds$$

for any $\phi(t) \in C(I_{\alpha_0}, \mathbb{R}^n)$.

By the claims below we know that T has a unique fixed point in $\mathcal{A}$, that is, there is a function $\bar{\phi} \in C(I_{\alpha_0}, \mathbb{R}^n)$ such that $(T\bar{\phi})(t) = \bar{\phi}(t)$. Hence, by Lemma 3.1, $\bar{\phi}$ is the solution to the initial value problem (4.1). This gives existence of the solution.

Now, if ϕ and ψ are two solutions of (4.1), defined on any subintervals J_ϕ and J_ψ about t_0 respectively. Let I_{α_0} ba the interval as above and $J = J_\phi \cap J_\psi \cap I_{\alpha_0}$. Then both ϕ and ψ are fixed points of the operator T_J corresponding to the interval J. But, the above argument shows that T_J is a contraction as well, and hence has a unique fixed point in $\mathcal{A}_J$. Since T_J has a unique fixed point, we must have $\phi = \psi$ on J. This implies the uniqueness of the solution in I_{α_0}. □

Claim 1. *T maps $\mathcal{A}$ into itself.*

Claim 2. *T is a contraction mapping on $\mathcal{A}$.*

Proof of Claim 1. Let $\phi \in \mathcal{A}$. Then, clearly $T\phi$ is a continuous map defined on all of I_{α_0}. Also, for $t \in I_{\alpha_0}$,

$$|(T\phi)(t) - x_0| \leq M|t - t_0| \leq M\alpha_0 < \beta,$$

so $T\phi \in \mathcal{A}$. □

Proof of Claim 2. Let $\phi, \psi \in \mathcal{A}$. The continuous function $|\phi(s) - \psi(s)|$ assumes its maximum at some point s_0 in I_{α_0}.

Let $t \geq t_0$. Then,

$$\begin{aligned}
|(T\phi)(t) - (T\psi)(t)| &= \left|\int_{t_0}^{t} f(s, \phi(s)) - f(s, \psi(s))ds\right| \\
&\leq \int_{t_0}^{t} K|\phi(s) - \psi(s)|ds \\
&\leq K|\phi(s_0) - \psi(s_0)|(t - t_0) \\
&\leq K|\phi - \psi|\alpha_0
\end{aligned}$$

The same inequality holds for $t < t_0$, so,

$$|T\phi - T\psi| \leq K\alpha_0|\phi - \psi|$$

Since, $K\alpha_0 < 1$, this shows that T is a contraction as required. □

4.2. Continuation of Solutions. Now we discuss continuation of solutions.

Consider the differential equation

$$\dot{x} = f(t, x). \tag{4.5}$$

Definition 4.2. *If ϕ is a solution of* (4.5) *defined on an interval I, we say that $\hat{\phi}$ is a* continuation *of ϕ or* extension *of ϕ if $\hat{\phi}$ is itself a solution of (4.5) defined on an interval $\hat{I}$ which properly contains I and $\hat{\phi}$ restricted to I equals ϕ.*

A solution is non-continuable *or* maximal *if no such extension exists; i.e., I is the maximal interval on which a solution to (4.5) exists.*

Lemma 4.3. *If D is an open subset of $\mathbb{R}\times\mathbb{R}^n$, and $f(t,x)$ is continuous and bounded on D, then any solution ϕ of (4.5) defined on an open interval (a,b) is such that the left and right limits $\phi(a_+)$ and $\phi(b_-)$ exist.*

If $f(b,\phi(b_-))$ is or can be defined so that $f(t,x)$ is continuous at $(b,\phi(b_-))$, then ϕ is a solution on the interval $(a,b]$ in the sense that the one-sided derivative $\lim_{t\to b_-} \dfrac{\phi(t)-\phi(b_-)}{t-b}$ exists and equals $f(b,\phi(b_-))$. A similar fact holds for the left endpoint a.

Proof. Let us first show that the left limit $\lim_{t\to b_-} \phi(t)$ exists.

Suppose that $|f(t,x)| \le M$ for all $(t,x) \in D$.

For any $t, t_0 \in (a,b)$, we have

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} f(s,\phi(s))ds.$$

Thus, for $t_1, t_2 \in (a,b)$,

$$|\phi(t_1) - \phi(t_2)| \le M|t_2 - t_1|.$$

which implies that as t_1, t_2 approach b from the left the norm $|\phi(t_1) - \phi(t_2)|$ approaches 0. This proves the existence of the desired left limit $\lim_{t\to b_-} \phi(t)$. A similar argument works for the right limit $\lim_{t\to a_+} \phi(t)$.

The last statement follows from the integral equation and the Fundamental Theorem of Calculus. □

Definition 4.3. *A* maximal solution *ϕ to a differential equation $\dot{x} = f(t,x)$ is a solution defined on an interval I such that there is no solution defined on an interval $\hat{I}$ which properly contains I.*

Theorem 4.4. *Suppose that $f(t,x)$ is defined, continuous, and locally Lipschitz in x in an open set $D \subseteq \mathbb{R}\times\mathbb{R}^n$, and ϕ is a solution defined* *on an interval I. Then, there is a maximal solution $\hat{\phi}$ on an interval $\hat{I}$ which contains I. As t approaches the boundary of $\hat{I}$, either $f(t, \hat{\phi}(t))$ becomes unbounded or $(t, \hat{\phi}(t))$ approaches the boundary of D.*

Proof. Let $\hat{I}$ be the union of all intervals containining I on which a solution exists. By uniqueness, they all patch together to give a maximal soution. Suppose $\hat{\phi}$ is this solution.

If $\hat{I}$ has a right boundary point, say b, and $f(t, \hat{\phi}(t))$ remains bounded as $t \to b_-$, then by the previous lemma, $\lim_{t \to b_-} \hat{\phi}(t) = x_0$ exists. If x_0 is in the interior of D, then patching $\hat{\phi}$ together with a solution to the IVP $\dot{x} = f(t, x), \;\; x(b) = x_0$, enables one to get a solution on an interval strictly larger than $\hat{I}$ which contradicts the defintion of $\hat{I}$. Thus, x_0 must be in the boundary of D. $\square$