5. Continuous Dependence of Solutions to Differential Equations on Parameters

We now want to investigate the dependence of solutions to differential equations on parameters.

Lemma 5.1 (Gronwall inequality). *Suppose $f(t)$, $a \leq t \leq b$, is a continuous non-negative real-valued function on the closed real interval $[a, b]$ such that there are positive constants K_1, K_2 such that for all $t \in [a, b]$,*

$$f(t) \leq K_1 + K_2 \int_a^t f(s)ds.$$

Then, for all $t \in [a, b]$,

$$f(t) \leq K_1 \exp[K_2(t-a)] \leq K_1 \exp[K_2(b-a)]$$

Proof. Let $U(t) = K_1 + K_2 \int_a^t f(s)ds$. Then, U is a strictly positive continuously differentiable function on $[a, b]$ with

$$U'(t) = K_2 f(t) \leq K_2 U(t)$$

for all t. Thus, $\frac{U'(t)}{U(t)} \leq K_2$. Integrating this inequality over the interval $[a, t]$ gives

$$\log U(t) - \log U(a) \leq K_2(t-a),$$

or

$$\log U(t) \leq \log U(a) + K_2(t-a),$$

and

$$f(t) \leq U(t) \leq U(a) \exp[K_2(t-a)] = K_1 \exp[K_2(t-a)].$$

□

LocCont

Theorem 5.2 (Local continuity of solutions on parameters). *Suppose $f(t, x, \lambda)$ is a continuous function defined in an open set $D \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k$. Suppose there are constants $M > 0$, $K > 0$ such that*

(1) *$| f(t, x, \lambda) | \leq M$ for all $(t, x, \lambda) \in D$,*
(2) *$| f(t, x, \lambda) - f(t, y, \lambda) | \leq K | x - y |$ for all $(t, x, \lambda), (t, y, \lambda) \in D$.*

Let $(t_0, x_0, \lambda_0) \in D$. Then, there are a positive number $\alpha > 0$ and a neighborhood V of (t_0, x_0, λ_0) such that for each $(u, y, \lambda) \in V$, the IVP

S5.IVP.uy

$$\dot{x} = f(t, x, \lambda), \qquad x(u) = y \tag{5.1}$$

has a unique solution $\phi(t, u, y, \lambda)$ defined on the interval $[u - \alpha, u + \alpha]$ and the function $\phi(t, u, y, \lambda)$ is a continuous function of the variables (t, u, y, λ) in $[t_0 - \alpha, t_0 + \alpha] \times V$.

Remark. *This result says that for all* (u, y, λ) *near* (t_0, x_0, λ_0)*, the solution to the IVP (5.1) is defined on the same sized interval (of length* 2α*) about the initial time* u *and the solution depends continuously on the initial time, value, and parameter.*

Proof. For any $(u, y, \lambda) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k$, $\alpha, \beta, \gamma > 0$, denote

$$\begin{aligned} I_\alpha(u) =& [u-\alpha, u+\alpha], \\ B_\beta(y) =& \{x \in \mathbb{R}^n : |x-y| \le \alpha\}, \\ C_\gamma(\tau) =& \{\gamma \in \mathbb{R}^n : |\gamma - \tau| \le \gamma\}. \end{aligned}$$

Take $\alpha_0, \beta_0, \gamma_0 > 0$ such that

$$V_0 := I_{\alpha_0}(t_0) \times B_{\beta_0}(x_0) \times C_{\gamma_0}(\lambda_0) \subset D.$$

Take $\beta = \beta_0/3$, and $\lambda = \lambda_0$. Then take $\alpha \in (0, \alpha_0/2)$ such that

$$\alpha M \le \beta, \qquad \alpha K < 1.$$

Then we let

$$V = I_\alpha(t_0) \times B_{2\beta}(x_0) \times C_\gamma(\lambda_0).$$

Clearly for any $(u, y, \tau) \in V$, we have that

$$I_\alpha(u) \times B_\beta(y) \times C_\gamma(\tau) \subset V_0.$$

By the E-U Theorem, we can prove that for $(u, y, \lambda) \in V'$, the IVP (5.1) has a unique solution $\phi(t, u, y, \lambda)$ defined on the interval $[u-\alpha, u+\alpha]$ with $\phi(t, u, y, \lambda) \in B_\beta(y)$. Then we take

$$V = I_\alpha(t_0) \times B_\beta(x_0) \times C_\gamma(\lambda_0) \subset V'.$$

It is easy to see that for any $(u, y, \lambda) \in V$, the solution of IVP (5.1) can be extended to the interval $[t_0 - \alpha, t_0 + \alpha]$. The details will be left as an exercise.

Now we prove that the solution $\phi(t, u, y, \lambda)$ is a continuous function on $I_\alpha(t_0) \times V$.

Let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that for any $(t, u, y, \lambda), (\bar{t}, \bar{u}, \bar{y}, \bar{\lambda}) \in I_\alpha(t_0) \times V$, if $|(t, u, y, \lambda) - \bar{t}, \bar{u}, \bar{y}, \bar{\lambda})| < \delta$, then

$$|\phi(t, u, y, \lambda) - \phi(\bar{t}, \bar{u}, \bar{y}, \bar{\lambda})| < \epsilon.$$

Take $\sigma > 0$ such that

$$2\alpha\sigma \cdot e^{2\alpha K} < \frac{\epsilon}{2},$$

then take $\delta > 0$ such that

$$(M+2)\delta \cdot e^{2\alpha K} < \frac{\epsilon}{2}$$

and for any $(u, y, \lambda), (u, y, \bar{\lambda}) \in V$ with $|\lambda - \bar{\lambda}| < \delta$,

flambda (5.2)
$$| f(u, y, \lambda) - f(u, y, \bar{\lambda}) | < \frac{\epsilon}{2}.$$

Now we take $(t, u, y, \lambda), (\bar{t}, \bar{u}, \bar{y}, \bar{\lambda}) \in I_\alpha(t_0) \times V$ with $|(t, u, y, \lambda) - (\bar{t}, \bar{u}, \bar{y}, \bar{\lambda})| < \delta$.

Note that $\phi(t, u, y, \lambda)$ satisfy the integral equation

$$\phi(t, u, y, \lambda) = y + \int_u^y f(s, \phi(s, u, y, \lambda), \lambda) ds.$$

Hence we have

fcont1 (5.3)
$$\begin{aligned} &\left|\phi(t, \bar{u}, \bar{y}, \bar{\lambda}) - \phi(\bar{t}, \bar{u}, \bar{y}, \bar{\lambda})\right| \\ =&\left|\int_{\bar{t}u}^{t} f(s, \phi(s, u, y, \lambda), \lambda) ds\right| \leq M|t - \bar{t}| < M\delta. \end{aligned}$$

Also, we have

fcont2 (5.4)
$$\begin{aligned} &|\phi(t, u, y, \lambda) - \phi(t, \bar{u}, \bar{y}, \bar{\lambda})| \\ =&\left|y + \int_u^t f(s, \phi(s, u, y, \lambda), \lambda) ds - \bar{y} - \int_{\bar{u}}^t f(s, \phi(s, \bar{u}, \bar{y}, \bar{\lambda}), \bar{\lambda}) ds\right| \\ \leq& \, | y - \bar{y} | + \left|\int_u^{\bar{u}} f(s, \phi(s, \bar{u}, \bar{y}, \bar{\lambda}), \bar{\lambda}) ds\right| \\ &\quad + \left|\int_u^t f(s, \phi(s, u, y, \lambda), \lambda) - f(s, \phi(s, \bar{u}, \bar{y}, \bar{\lambda}), \bar{\lambda}) ds\right| \end{aligned}$$

Obviously,

fcont3 (5.5)
$$| y - \bar{y} | \leq \delta.$$

fcont4 (5.6)
$$\left|\int_u^{\bar{u}} f(s, \phi(s, \bar{u}, \bar{y}, \bar{\lambda}), \bar{\lambda}) ds\right| \leq M|\bar{u} - u| < M\delta.$$

Now we estimate the last integral in (fcont2 5.4). Note that $t, u \in I_\alpha(t_0)$, we have $|t - u| \leq 2\alpha$. Without loss generality we may assume that

$t > u$. Hence, by (5.2) and the assumption that f is Lipschitz in $\mathbb{R}^n$,

fcont5

$$\begin{aligned} &\Big|\int_u^t f(s,\phi(s,u,y,\lambda),\lambda) - f(s,\phi(s,\bar{u},\bar{y},\bar{\lambda}),\bar{\lambda})ds\Big| \\ \le &\int_u^t \mid f(s,\phi(s,u,y,\lambda),\lambda) - f(s,\phi(s,u,y,\lambda),\bar{\lambda}) \mid ds \\ + &\int_u^t \mid f(s,\phi(s,u,y,\lambda),\bar{\lambda}) - f(s,\phi(s,\bar{u},\bar{y},\bar{\lambda}),\bar{\lambda}) \mid ds \\ \le & 2\alpha\sigma + \int_u^t K \mid \phi(s,u,y,\lambda) - \phi(s,\bar{u},\bar{y},\bar{\lambda}) \mid ds. \end{aligned} \tag{5.7}$$

By (5.4)-(5.7), we get

$$\begin{aligned} &\mid \phi(t,u,y,\lambda) - \phi(t,\bar{u},\bar{y},\bar{\lambda}) \mid \\ \le &((M+1)\delta + 2\alpha\sigma) + \int_u^t K \mid \phi(s,u,y,\lambda) - \phi(s,\bar{u},\bar{y},\bar{\lambda}) \mid ds. \end{aligned}$$

By the Gronwall inequality, we get

$$\mid \phi(t,u,y,\lambda) - \phi(t,\bar{u},\bar{y},\bar{\lambda}) \mid \le [(M+1)\delta + 2\alpha\sigma]\exp[K\cdot 2\alpha].$$

So by (5.3), and the choice of σ and δ,

$$\begin{aligned} &\mid \phi(t,u,y,\lambda) - \phi(\bar{t},\bar{u},\bar{y},\bar{\lambda}) \mid \\ \le &[(M+1)\delta + 2\alpha\sigma]\exp[K\cdot 2\alpha] + M\delta \\ \le &(M+2)\delta e^{2\alpha K} + 2\alpha\sigma e^{2\alpha K} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This gives the desired continuity statement. □

Recall that $\phi(t,u,y,\lambda)$ is the solution of the IVP

$$\dot{x} = f(t,x,\lambda), \qquad x(u) = y.$$

LGlCont1 **Lemma 5.3.** *Suppose $\phi(t,u,y,\lambda)$ and $\phi(t,\bar{u},\bar{y},\lambda)$ are defined on intervals I and $\bar{I}$ respectively. If there exists $c \in I \cap \bar{I}$ such that*

$$\phi(c,u,y,\lambda) = \phi(c,\bar{u},\bar{y},\lambda),$$

then $\phi(t,u,y,\lambda)$ can be extended to $I \cup \bar{I}$ such that

$$\phi(t,u,y,\lambda) = \phi(t,\bar{u},\bar{y},\lambda)$$

for all $t \in \bar{I}$.

Proof. It is directly from uniqueness. □

Let $t \in \mathbb{R}$. Suppose $\phi(s,u,y,\lambda)$ is defined for any $s \in [u, u+t]$ if $t \geq 0$ and $s \in [u+t, u]$ if $t < 0$. Denote

$$\Phi^t(u,y,\lambda) = (u+t, \phi(u+t,u,y,\lambda), \lambda).$$

If Φ^t is defined for any element (u,y,λ) in a set $V \subset \mathbb{R} \times \mathbb{R}^n \times R^k$, then we denote

$$\Phi^t(u,y,\lambda) = \{\Phi^t(u,y,\lambda) : \quad (u,y,\lambda) \in V\}.$$

LGlCont2 **Lemma 5.4.** *(i) For any $r, s \in \mathbb{R}$, $\Phi^r \circ \Phi^s = \Phi^{r+s}$, whenever they are defined.*

(ii) Suppose Φ^t is defined on an set V. Then Φ^t is a homeomorphism fron V to its image.

Proof. (i) Applying Lemma 5.3 (LGlCont1) with $\bar{u} = u+s$ and $\bar{y} = \phi(u+s,u,y,\lambda)$ we have

$$\phi(u+r+s, u+s, \phi(u+s,u,y,\lambda), \lambda) = \phi(u+r+s,u,y,\lambda),$$

because the equation holds for $c = u+s$. Hence,

$$\begin{aligned}\Phi^r \circ \Phi^s(u,y,\lambda) =& \Phi^r(u+s, \phi(u+s,u,y,\lambda), \lambda) \\ =& (u+r+s, \phi(u+r+s,u,y,\lambda), \lambda) = \Phi^{r+s}(u,y,\lambda).\end{aligned}$$

(ii) This is from part (ii) and the local continuity of solutions on parameters. $\square$

GlCont **Theorem 5.5** (Global continuity of solutions on parameters)**.** *Suppose $f(t,x,\lambda)$ is continuous and locally Lipschitz in x in an open set $D \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k$. If $\phi(t,a,x_0,\lambda_0)$ is a solution of the IVP*

S5.IVP.ax
$$\dot{x} = f(t,x,\lambda_0), \qquad x(a) = x_0, \tag{5.8}$$

which is defined on the closed interval $[a,b]$ and $(t, \phi(t,a,x_0,\lambda_0), \lambda_0) \in D$ for $t \in [a,b]$, then there is a neighborhood V of (a,x_0,λ_0) in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k$ such that, for $(u,y,\lambda) \in V$, the IVP

S5.IVP.uy2
$$\dot{x} = f(t,x,\lambda), \qquad x(u) = y \tag{5.9}$$

also has a solution defined on the interval $[u,b]$.

Moreover, the function $\phi(t,u,y,\lambda)$ is continuous on $[u,b] \times V$.

Proof. Since $[a,b]$ is a compact set and $\phi(t,a,x_0,\lambda_0)$ is continuous, the set

$$A = \{(t, \phi(t,a,x_0,\lambda_0), \lambda_0) : t \in [a,b]\}$$

is a compact subset of D. Therefore, there are constants $M > 0, K > 0$ for which the conditions on boundedness and Lipschitzness of Theorem 5.2 (LocCont) hold with these constants throughout a neighborhood $D' \subset D$

of A. Hence, by Theorem 5.2, for each $s \in [a, b]$, there exists a constance $\alpha_s > 0$ and a neighborhood V_s of $(s, \phi(s, a, x_0, \lambda_0), \lambda_0)$ such that for any $(u, y, \lambda) \in V^*$, the IVP (5.9) has a solution $\phi(t, u, y, \lambda)$ defined and continuous on $[s - \alpha_s, s + \alpha_s] \times V_s$.

For any $\beta > 0$, set

$$U_\beta = \{(u, y, \lambda) \in V_a : (5.9) \text{ has a solution defined on } [u, \beta]\}$$

Clearly, $(a, x_0, \lambda_0) \in U_b$, and if $\beta \leq \alpha$, then $U_\beta \supset U_a$. Also, by Theorem 5.2, there is $\alpha > 0$ and a neighborhood V of (a, x_0, λ_0) such that $V \subset V_\alpha$.

Take

$$\beta^* = \sup\{\beta : U_\beta \text{ contains a neighborhood of } (a, x_0, \lambda_0)\}.$$

If $\beta^* > b$, then we get that U_b contains a neighborhood of (a, x_0, λ_0) and this is what we need.

Suppose $\beta^* \leq b$.

Denote $\beta' = \beta^* - \alpha_{\beta^*}/2$, and then let $x' = \phi(\beta', a, x_0, \lambda_0)$, and $x^* = \phi(\beta^*, a, x_0, \lambda_0)$.

Since V_{β^*} is a neighborhood of $(\beta^*, x^*, \lambda_0)$, $\Phi^{-\alpha^*/2}(V_{\beta^*})$ is a neighborhood of (β', x', λ_0).

Since $U_{\beta'}$ is a neighborhood of (a, x_0, λ_0), $\Phi^{\beta'-a}(U_{\beta'})$ is a neighborhood of (β', x', λ_0).

Take $V' = \Phi^{-\alpha^*/2}(V_{\beta^*}) \cap \Phi^{\beta'-a}(U')$, then V' is a neighborhood of (β', x', λ_0). Let $V = \Phi^{-\beta'+a}(V')$.

By definition of Φ^t, for any $(u', y', \lambda) \in V'$, there exists $(u, y, \lambda) \in V \subset U_{\beta'}$ such that $\phi(u', u, y, \lambda) = (u', y', \lambda')$ and the solution is defined on $[u, u']$. Also, there exists $(u^*, y^*, \lambda) \in V_{\beta^*}$ such that $\phi(u'u^*, y^*, \lambda) = (u', y', \lambda')$ and the solution is defined on $[u', u^* + \alpha*]$. Since $\phi(u', u, y, \lambda) = \phi(u', u^*, y^*, \lambda)$, by Lemma 5.3, $\phi(t, u, y, \lambda)$ is defined on $[u, u^* + \alpha*]$.

We may choose V_{β^*} small enough such that for all $(u^*, y^*, \lambda) \in V_{\beta^*}$, $|u^* - \beta^*| \leq \alpha^*/2$, and hence $u^* + \alpha^* > \beta^* + \alpha^*/2$. So we conclude that for any $(u, y, \lambda) \in V$, the solution $\phi(t, u, y, \lambda)$ is defined on $[u, \beta^* + \alpha^*/2]$. This is a contradiciton. Hence, we get $\beta^* > b$.

Continuity of $\phi(t, u, y, \lambda)$ on $[u, b] \times V$ follows from local continuity. $\square$

We have proved that solutions to differential equations depend continuously on initial values and parameters.

Now we wish to investigate the smooth dependence of solutions in systems which depend smoothly on initial values and parameters.

Lemma 5.6 (Hadamard's Lemma)**.** *Suppose $g : D \to \mathbb{R}^n$ is a C^1 function, where $D \subset \mathbb{R}^m$ is an open set. For any points $x, x_0 \in D$ with*

$tx + (1-t)x_0 \in D$ for any $t \in [0,1]$, there is a function $G(x, x_0)$ such that

$$g(x) - g(x_0) = G(x, x_0)(x - x_0)$$

and

$$G(x_0, x_0) = \lim_{x \to x_0} G(x, x_0) = \frac{\partial g}{\partial x}(x_0).$$

Proof. Note that

$$g(x) - g(x_0) = \int_0^1 \frac{\partial g}{\partial s}(x_0 + s(x - x_0))ds = \int_0^1 \frac{\partial g}{\partial x}(x_0 + s(x - x_0)) \cdot (x - x_0)ds.$$

So we can take

$$G(x, x_0) = \int_0^1 \frac{\partial g}{\partial x}(x_0 + s(x - x_0))ds.$$

Since $\frac{\partial g}{\partial x}$ is continuous, so is $G(x, x_0)$. Also, the second equality follows from the definition of $G(x, x_0)$. □

Theorem 5.7 (Differentiability of solutions on parameters)**.** *Suppose that $f(t, x, \lambda)$ is a C^1 function of the variables (t, x, λ) in an open set $D \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k$. For $(t_0, x_0, \lambda_0) \in D$, let $\phi(t, t_0, x_0, \lambda_0)$ be the solution of the initial value problem*

S5.IVP.tx0 (5.10)
$$\dot{x} = f(t, x, \lambda_0), \qquad x(t_0) = x_0.$$

Then, the solution $\phi(t, t_0, x_0, \lambda_0)$ is a C^1 function of the variables (t, t_0, x_0, λ_0).

Moreover, the partial derivatives $\frac{\partial \phi}{\partial t_0}(t, t_0, x_0, \lambda_0)$ and $\frac{\partial \phi}{\partial x_0}(t, t_0, x_0, \lambda_0)$ of the solution with respect to the initial time t_0 and initial position x_0 satisfies the initial value problem

ivp_3 (5.11)
$$\dot{Z} = \frac{\partial f}{\partial x_0}(t, \phi(t, t_0, x_0, \lambda_0), \lambda_0) \cdot Z,$$

and

ivp_4 (5.12)
$$\begin{aligned} \frac{\partial \phi}{\partial t_0}(t, t_0, x_0, \lambda_0)\Big|_{t=t_0} &= -f(t_0, x_0, \lambda_0) \\ \frac{\partial \phi}{\partial x_0}(t, t_0, x_0, \lambda_0)\Big|_{t=t_0} &= \mathrm{id} \end{aligned}$$

respectively, and the partial derivative $\frac{\partial \phi}{\partial \lambda}(t, t_0, x_0, \lambda_0)$ *of the solution with respect to parameter* λ_0 *satisfies the initial value problem*

$$\begin{aligned} \dot{Z} &= \frac{\partial f}{\partial x_0}(t, \phi(t, t_0, x_0, \lambda_0), \lambda_0) \cdot Z + \frac{\partial f}{\partial \lambda_0}(t, \phi(t, t_0, x_0, \lambda_0), \lambda_0), \\ &\frac{\partial \phi}{\partial \lambda_0}(t, t_0, x_0, \lambda_0)\Big|_{t=t_0} = 0. \end{aligned} \tag{5.13}$$

Proof. We only prove for $\frac{\partial \phi}{\partial x_0}$.

For the sake of simplifying the notation we drop λ_0 in the formulas.

Assume $x' = x_0 + re_i$, where e_i is the unit vector in the ith direction in $\mathbb{R}^n$. Define

$$\psi(t, t_0, x, x_0) = \frac{1}{r}\big(\phi(t, t_0, x) - \phi(t, t_0, x_0)\big).$$

By the Hadamard's lemma, we get

$$\begin{aligned} \dot{\psi}(t, t_0, x, x_0) &= \frac{1}{r}\big(f(t, \phi(t, t_0, x)) - f(t, \phi(t, t_0, x_0))\big) \\ =&\frac{1}{r}F(t, \phi(t, t_0, x), \phi(t, t_0, x_0))\big(\phi(t, t_0, x) - \phi(t, t_0, x_0)\big) \end{aligned}$$

for some function F. It means that ψ satisfies the initial value problem

$$\dot{Z} = F(t, \phi(t, t_0, x), \phi(t, t_0, x_0)) \cdot Z, \qquad Z(t_0) = \frac{1}{r}(x - x_0) = e_i.$$

Note that by the Hadamard's lemma,

$$\lim_{x \to x_0} F(t, \phi(t, t_0, x), \phi(t, t_0, x_0) = \frac{\partial f}{\partial x}(t, \phi(t, t_0, x_0)).$$

Suppose $\psi(t, t_0, e_i)$ is a solution of the initial value problem

$$\dot{Z} = \frac{\partial f}{\partial x}(t, \phi(t, t_0, x_0)) \cdot Z, \qquad Z(t_0) = e_i.$$

Then by continuity of solution of differential equation on *parameters*,

$$\psi(t, t_0, e_i) = \lim_{x \to x_0} \psi(t, t_0, x, x_0) = \frac{\partial \phi}{\partial x_0}(t, \phi(t, t_0, x_0)),$$

which is what we need. □

Remark. *The differential equation in* [ivp_3] *(5.11) and* [ivp_4] *(5.12) (without the* λ *parameter) is usually called the variational equation of* [S5.IVP.tx0] *(5.10).*