6. General Properties of Autonomous Systems

Consider an *autonomous system*

$$\dot{x} = f(x) \tag{6.1}$$

with $f : D \to \mathbb{R}^n$ a locally Lipschitz map from an open set $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

We also say that such a function f is a *vector field* in D. Thus, being given a vector field in D is the same as being given an autonomous differential equation in D.

According to the existence and uniqueness theorem, for each point $x \in D$, there is a unique solution $\phi(t)$ with $\phi(0) = x$ defined in an interval I about 0 in $\mathbb{R}$. Let us denote this function by $\phi(t, x)$. We claim

$$\phi(t + s, x) = \phi(t, \phi(s, x))$$

for $t, s, t + s \in I$.

Indeed, the function $t \to \phi(t + s, x)$ is a solution to the differential equation $\dot{x} = f(x)$ and $\phi(0+s, x) = \phi(s, x)$. By uniqueness of solutions, $\phi(t + s, x) = \phi(t, \phi(s, x))$.

This property is called the *local flow property* of autonomous systems. The map $t \to \phi(t, x)$, $t \in I$ then defines a C^1 curve in D.

If, in addition, for each $x \in D$, the solution $\phi(t, x)$ is defined for all time, it is called a *flow* in D. We then have $\phi(t + s, x) = \phi(t, \phi(s, x))$ for all $s, t \in \mathbb{R}$ and $x \in D$. Writing ϕ_t for the mapping $\phi(t, \cdot)$, we can write this last property as

$$\phi_{t+s} = \phi_t \circ \phi_s.$$

Definition 6.1. *Let D_1, D_2 be two non-empty open sets in $\mathbb{R}^n$. Let r be a positive integer. A C^r map $g : D_1 \to D_2$ is called a C^r* diffeomorphism *if g is one-to-one, onto, and its inverse mapping $g^{-1} : D_2 \to D_1$, is also C^r.*

By the inverse function theorem, a C^r map $g : D_1 \to D_2$ is a C^r diffeomorphism if and only if it is 1-1, onto, and, for each $x \in D_1$, the derivative of g at x is a linear isomorphism of $\mathbb{R}^n$.

The set $\mathrm{Diff}^r(D)$ of C^r diffeomorphisms from D to itself is a non-commutative group with the composition operation as group product

$$g_1 \cdot g_2 = g_1 \circ g_2.$$

The identity of this group is simply the identity transformation.

A flow $\{\phi_t\}_{t\in\mathbb{R}}$ form a one parameter commutative group. The group is isomorphic to the group $(\mathbb{R}, +)$. Sinmlarly, for any diffoemorphism

g of an open set D, the set $\{g^n : n \in \mathbf{Z}\}$ ia also a commutative group isomorphic to $(\mathbf{Z}, +)$.

Exercise. Recall that a *topological group* is a triple $(G, \mathcal{T}, \cdot)$ in which $(G, \mathcal{T})$ is a topological space, $(G, \cdot)$ is a group (i.e., $\cdot$ is an associative operation with an identity and each element h has an inverse h^{-1}), and the mapping $(g, h) \to g \cdot h^{-1}$ from the product space $G \times G$ to G is continuous. Sometimes we say that the group G is a topological group and we suppress writing the explicit topology and group operation. Given the set $\text{Diff}^r(D)$ the topology of uniform C^r convergence on compact subsets of D, this topology makes the group $(\text{Diff}^r(D), \circ)$ into a topological group, where "$\circ$" is composition of maps in $\text{Diff}^r(D)$.

Given a vector field (4), and a point $x \in D$, we call the maximal soltution curve $t \to \phi(t, x)$ the *orbit* or *path* through x. Sometimes we also call the image set $\{\phi(t, x)\}$ of a *maximal solution* through x, the *orbit* through x.

We also denote by $\mathcal{O}(x)$ the orbit through x, that is,

$$\mathcal{O}(x) = \{\phi(t, x) : t \in \mathbb{R}\}.$$

Lemma 6.1. *Any two orbits are distinct or identical.*

Proof. Suppose $y \in \mathcal{O}(x)$ and we show that $\mathcal{O}(y) = \mathcal{O}(x)$.

We have $y = \phi(t, x)$ for some $t \in \mathbb{R}$. Hence for any $s \in \mathbb{R}$, $\phi(s, y) = \phi(t+s, x) \in \mathcal{O}(x)$. So $\mathcal{O}(y) \subset \mathcal{O}(x)$. Similarly by that fact $x = \phi(-t, y)$ we have $\mathcal{O}(x) \subset \mathcal{O}(y)$. $\square$

Lemma 6.2. *Any orbit is a point, a circle, or a 1-1 continuous image of an open interval (including the whole line).*

Proof. Consider an orbit $\{\phi(t, x) : t \in I\}$, where I is an open interval.

If for any $s, t \in I$ with $s \neq t$, $\phi(s, x) \neq \phi(t, x)$, then the orbit is a 1-1 continuous image of I.

Suppose $\phi(s, x) = \phi(t, x)$ for some $s, t \in I$ with $s < t$. Then for any $r \in [0, t - s]$, $\phi(s - r, x) = \phi(t - r, x)$ and $\phi(t + r, x) = \phi(s + r, x)$ are well defined and therefore $\phi(\cdot, x)$ is defined on the whole real line by induction. Moreover, for any $r \in \mathbb{R}$, we have $\phi(s + r, x) = \phi(t + r, x)$. In particular, $x = \phi(0, x) = \phi(t - s, x) = \phi(\tau, x)$, where $\tau = t - s$.

If τ is the smallest positive number such that $\phi(\tau, x) = x$, then the orbit through x is a circle.

If there is sequence of positive number $\{t_n\}$ with $t_n \to 0$ such that $\phi(t_n, x) = x$, then we have $\phi(kt_n, x) = x$ for any $k \in \mathbb{Z}$. It means that there is a dense subset S, $S = \{kt_n : k \in \mathbb{Z}, n \in \mathbb{N}\}$, in $\mathbb{R}$ such that $\phi(s, x) = x$ for any $s \in S$. By continuity we get that $\phi(t, x) = x$ for any $t \in \mathbb{R}$. So the orbit is a point. $\square$

The solution whose orbit is a topological circle is called a *periodic solution*. From that proof above we see that a periodic solution $\phi(t, x)$ is one for which there is a real $\tau > 0$ such that $\phi(\tau, x) = x$ and $\phi(s, x) \neq x$ for $0 < s < \tau$. The number τ is called the *period* of the periodic solution. Sometimes periodic solutions are called *periodic orbits* or *closed orbits*.

The solution whose orbit is a point is called a *critical solution*, or a *critical point* of $\dot{x} = f(x)$. Sometimes it is called an *equilibrium* or a *stationary point*.

It is easy to see the following.

Lemma 6.3. *A point $x_0 \in D$ is a critical point of a vector field f if and only if $f(x_0) = 0$.*

Since any solution may be extended to a maximal solution, from now on, unless explicitly stated otherwise, every solution will be assumed to be maximal.