7. Limit Sets

Consider an autonomous system of the form $\dot{x} = f(x)$ for which solutions are defined for all time in an open set D.

Definition 7.1. *Let $x \in D$.*

The ω-limit set *of x, denoted by $\omega(x)$, is the set of points y such that there is a sequence $t_1 < t_2 < \dots$ with $t_i \to +\infty$ and $\phi(t_i, x) \to y$ as $i \to \infty$.*

The α-limit set *of x, denoted by $\alpha(x)$, is the set of points y such that there is a sequence $t_1 > t_2 > \dots$ with $t_i \to -\infty$ and $\phi(t_i, x) \to y$ as $i \to \infty$.*

For any subset $\Lambda \subset D$ and $t \in \mathbb{R}$, denote $\phi(t, \Lambda) = \{\phi(t, x) : x \in \Lambda\}$.

Definition 7.2. *A subset $\Lambda \subset D$ is* invariant *for the differential equation $\dot{x} = f(x)$ or for the vector field f in D if for any $t \in \mathbb{R}$, $\phi(t, \Lambda) \subseteq \Lambda$, where $\phi(t, x)$ is a solution to $\dot{x} = f(x)$ with $\phi(0, x) = x$.*

Note that if $\phi(t, \Lambda) \subseteq \Lambda$ for any $t \in \mathbb{R}$, then we also have $\phi(-t, \Lambda) \subseteq \Lambda$ and therefore $\Lambda = \phi(t, \phi(-t, \Lambda)) \subseteq \phi(t, \Lambda)$. So we can replace "$\subseteq$" by "$=$" in the definition.

Any orbit is an invariant set by the definition.

Lemma 7.1. *For any $x \in D$, $\omega(x)$ and $\alpha(x)$ are closed invariant subsets.*

Proof. Invariance) We prove $\phi(t, \omega(x)) \subseteq \omega(x)$ for any $t \in \mathbb{R}$. Let $y \in \omega(x)$. Then we can take $t_1 < t_2 < \dots$ such that $t_i \to -\infty$ and $\phi(t_i, x) \to y$ as $i \to \infty$. Hence, by continuity we have $t_1 + t < t_2 + t < \dots$ such that $t_i + t \to \infty$ and $\phi(t_i + t, x) = \phi(t, \phi(t_i, x)) \to \phi(t, y)$ as $i \to \infty$. So $\phi(t, y) \in \omega(x)$, which implies $\phi(t, \omega(x)) \subseteq \omega(x)$.

(Closeness) Suppose $\{y_k\} \subset \omega(x)$ such that $y_k \to y_0$ as $k \to \infty$. For each $k > 0$, we can choose $t_k > 0$ inductively such that $d(\phi(t_k, x), y_k) \leq d(y_k, y_0)$ and $t_k > \min\{t_{k-1}, k\}$. We get a sequence $t_1 < t_2 < \dots$ with $t_k \to \infty$. Since

$$d(\phi(t_k, x,), y_0) \leq d(\phi(t_k, x,), y_k) + d(y_k, y_0) \leq 2d(y_k, y_0) \to 0,$$

we have $\phi(t_k, x,) \to y_0$ as $k \to \infty$. So $y_0 \in \omega(x)$, and therefore $\omega(x)$ is a closed subset. □

Fact 7.2. *Suppose f is a C^1 vector field in $D \subset \mathbb{R}^n$ and $x \in D$ has the property that the orbit $\phi(t, x)$ of x remains in a compact subset f of D for $t \geq 0$. Then, $\omega(x)$ is a compact invariant connected subset of f.*

Definition 7.3. *A point $x \in D$ is a* nonwandering point *of the flow ϕ_t if for any neighborhood U of x and any $T > 0$, there exists $t > T$ such that*

$$\phi_t(U) \cap U \neq \emptyset.$$

The set of all nonwandering points is called the nonwandering set

It can be proved that a point $x \in D$ is a *nonwandering point* if for any neighborhood U of x, there exists $t > 1$ such that $\phi_t(U) \cap U \neq \emptyset$.

Clearly, for any $x \in D$, any point $y \in D$ in an α-limit set $\alpha(x)$ or ω-limit set $\omega(x)$ is nonwandering point.

Lemma 7.3. *Nonwandering set is closed invariant set.*

Proof. Invariance is clear. The set is closed because by definition its complement is opan. □

Definition 7.4. *A subset set is called a* minimal set *of a flow if it is a nonempty closed invariant set that does not contain any closed invariant proper subset.*

Proposition 7.4. *Any compact invariant set contains a minimal set.*

Proof. Let K be a compact invariant set. Let $\mathcal{S}$ be the collection of all nonempty invariant compact subset A of K. Define a partial order "$\prec$" in $\mathcal{S}$ by $A \prec B$ if $A \subset B$. Then by Zorn's lemma, every linearly ordered subsets $\cdots \prec A_i \prec A_{i-1} \prec \dots$ has a least element Σ. In fact, Σ is the intersection of the set $\{A_i\}$, and is nonempty by Cantor intersection theorem. Σ is a minimal set. □

Lemma 7.5. *A compact set Σ is minimal if and only if for any $x \in \Sigma$, $\overline{\mathcal{O}(x)} = \Sigma$.*

Proof. "$\Longrightarrow$": This is because $\overline{\mathcal{O}(x)}$ is a nonempty colsed invariant set contained in Σ.

"$\Longleftarrow$": If Σ is not a minimal set, then there is a nonempty closed invariant subset Σ_1 properly contained in Σ, then for any $x \in \Sigma_1$, $\overline{\mathcal{O}(x)} \subset \Sigma_1 \neq \Sigma$. □

Example. A critical point or periodic orbit is a minimal set.

It is remarkable fact that in the plane for a C^1 autonomous vector field, there are no other minimal sets. On the other hand, in $\mathbb{R}^n, n > 2$, there are many examples of non-trivial minimal sets.