8. Structure of autonomous differential equations near a non-critical point

8.1. Vector Fields as Differential operators. Recall that an autonomous differential equation $\dot{x} = f(x)$ is given by a function $f : D \to \mathbb{R}^n$ from a domain D in $\mathbb{R}^n$. Suppose that f is C^k for $k \geq 1$. Let $C^k(D, \mathbb{R})$ be the space of C^k real-valued functions defined on D. We can use f to define an operator $\mathcal{L}_f$ from $C^{k+1}(D, \mathbb{R})$ to $C^k(D, \mathbb{R})$ in the following way.

For $x \in D$, let $\phi(t, x)$ be the solution to $\dot{x} = f(x)$, $\phi(0, x) = x$. For, $\psi \in C^{k+1}(D, \mathbb{R})$, let

$$(\mathcal{L}_f \psi)(x) = \frac{d}{dt}\psi(\phi(t, x))\big|_{t=0}. \tag{8.1}$$

Fact 8.1. *$\mathcal{L}$ is a mapping from $C^{k+1}(D, \mathbb{R})$ to $C^k(D, \mathbb{R})$ satisfying the following two properties.*

(1) *(Linearity). $\mathcal{L}$ is a linear mapping; i.e., for any $\psi, \eta \in C^{k+1}(D, \mathbb{R})$, scalars α, β*

$$\mathcal{L}_f(\alpha\psi + \beta\eta) = \alpha\mathcal{L}_f(\psi) + \beta\mathcal{L}_f(\eta).$$

(2) *(Derivation). For $\psi, \eta \in C^{k+1}(D, \mathbb{R})$,*

$$\mathcal{L}_f(\psi \cdot \eta) = \mathcal{L}_f(\psi) \cdot \eta + \psi\mathcal{L}_f(\eta).$$

The operator $\mathcal{L}_f$ is called the *Lie derivative operator*. It maps C^{k+1} functions to C^k functions.

Let $\pi_i : x \to x_i$ be the projection of a vector onto its i-th coordinate as a function on $\mathbb{R}^n$.

Fact 8.2. *Denote $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. Then*

(1) *The value of the function $\mathcal{L}_f(\psi)$ can be computed by the formula*

$$\mathcal{L}_f(\psi)(x) = \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i}(x) \cdot f_i(x). \tag{8.2}$$

(2) $\mathcal{L}_f(\pi_i) = f_i$.

(3) *The function ψ is constant along solution curves of $\dot{x} = f(x)$ if and only if $\mathcal{L}_f(\psi)$ is the zero function in D.*

Proof. (1) This is because by (8.1),

$$(\mathcal{L}_f\psi)(x) = \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i}(\phi(t, x))\big|_{t=0} \frac{d(\pi_i \circ \phi)(t, x)}{dt}\big|_{t=0}.$$

Since $\phi(t,x)\big|_{t=0} = x$, $\pi_i \circ \phi = \phi_i$, and $\dot{\phi}_i(t,x) = f_i(\phi(t,x))$, we get (8.2).

(2) It follow from the fact that $\dfrac{\partial \pi_j}{\partial x_i}(x) = 1$ if $j = i$ and 0 otherwise.

(3) It follows directly from the definiton of $\mathcal{L}_f$ given in (8.1). □

By part (2), the operator $\mathcal{L}_f$ and the vector field f completely determine each other, and we can think of vector fields as differential operators on real-valued functions or as assignments of vectors at each point in a domain D. Hence, we will often identify an autonomous differential equation $\dot{x} = f(x)$ with the vector field f and with the operator $\mathcal{L}_f$.

Let e_i be the unit vector in $\mathbb{R}^n$ whose i-th coordinate is 1 and whose other coordinates are 0. It is common to write $\frac{\partial}{\partial x_i}$ for the operator $\mathcal{L}_f$ where $f(x) = e_i$ is the constant vector field whose value at each x is e_i. In this sense, we can write

$$f(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}.$$

This means that given a function $f : D \to \mathbb{R}^n$, with $f = (f_1, \ldots, f_n)$, we get any one of three objects: the system of differential equations

$$\dot{x}_i = f_i(x), \quad i = 1, \ldots, n,$$

the vector field

$$x \to f(x), \quad x \in D,$$

and the operator

$$\psi \to \mathcal{L}_f(\psi).$$

8.2. The flow-box theorem.

Definition 8.1. *Suppose f is a vector field in the domain $D \subset \mathbb{R}^n$. Let $\rho : D \to D'$ be a smooth change of coordinates from D to the domain D'. Then, ρ maps the vector field f to the new vector field $\rho_\star f$ defined by*

$$\rho_\star(f)(y) = D\rho_{\rho^{-1}y}(f(\rho^{-1}y)).$$

Thus, we can write $\rho_\star = D\rho \circ f \circ \rho^{-1}$ as vector valued funtions.

Theorem 8.3 (Flow-box theorem, path-cylinder theorem)**.** *Let $k \geq 1$. Suppose f is a C^k vector field in a domain D and x_0 is a point in D such that $f(x_0) \neq 0$. Then there is a C^k change of coordinates ρ from a neighborhood U of 0 in $\mathbb{R}^n$ to a neighborhood V of x_0 such that ρ carries solutions of the constant vector field $\dfrac{\partial}{\partial x_1}$ onto those of $\dot{x} = f(x)$.*

Proof. Since $f(x_0) \neq 0$, we may consider $f(x_0)$ as a vector attached to the origin 0 in $\mathbb{R}^n$ and pick non-zero unit vectors $\xi_2, \xi_3, \ldots \xi_n$ so that the vectors $f(x_0), \xi_2, \xi_3, \ldots, \xi_n$ are linearly independent. Let $\tilde{H}$ be the subspace of $\mathbb{R}^n$ spanned by the vectors $\xi_i, i \geq 2$. The affine subspace $H = x_0 + \tilde{H}$ is then transverse to the vector field $f(x_0)$ at x_0.

By the local continuity of solutions to $\dot{x} = f(x)$ on initial conditions and the continuity of f, there are a neighborhood V_1 of x_0 in H and an interval I about 0 in $\mathbb{R}$ such that for any $x \in V_1$, (i) $\phi(t, x)$ is defined on all of I and (ii) $\phi(t, x)$ meets H only for $t = 0$.

We define a mapping $\rho : \tilde{H} \to H$ by $\eta(\tilde{y}) = x_0 + \sum_j y_j \xi_j$ if $\tilde{y} = (y_2, \ldots, y_n) = \sum_j y_j \xi_j \in \tilde{H}$. Let $U_1 = \rho^{-1}(V_1)$, and $U_0 = I \times U_1$. Then we extend the mapping ρ to U_0 by

$$\rho(y) = \rho(y_1, \tilde{y}) = \phi(y_1, \rho(\tilde{y})), \tag{8.3}$$

if $y = (y_1, \tilde{y}) \in U_0$. We claim that this transformation ρ give the required change of coordinates.

First, it is obvious that ρ is a one to one map. Also, ρ is a C^k mapping of the variables $(y_1, \tilde{y})$.

To prove that ρ is a change of coordinates, it suffices to show that its Jacobian determinant at 0 is not zero so that we can use the implicit function theorem to get that ρ^{-1} is also a C^k mapping.

Note that at $(y_1, \tilde{y}) = 0$, the first column of the Jacobian matrix of ρ, $\dfrac{\partial \rho}{\partial y_1}$ is just $f(x_0)$. Since $\phi(0, \eta(\tilde{y})) = \eta(\tilde{y})$, it follows $\dfrac{\partial \rho}{\partial y_j} = \dfrac{\partial \eta}{\partial y_j} = \xi_j$ at 0 for $j = 2, \ldots, n$. By the choice of the $\xi_j's$, $f(x_0), \xi_2, \xi_3, \ldots, \xi_n$ are linearly independent. Thus, the required Jacobian determinant is not zero. By continuity, the Jacobian determinant is not zero in a neighborhood $U \subset U_0$. Denote $V = \rho(U)$. By the Inverse Function Theorem we get that the inverse ρ^{-1} is defined and is C^k.

Finally, we have to show that the mapping ρ carries solutions to $\dfrac{\partial}{\partial x_1}$ to those of f. Denote the solution of the vector fild $\dfrac{\partial}{\partial x_1}$ by $\psi(t, x)$. Then for any $y = (y_1, \tilde{y})$,

$$\psi(t, y) = \psi(t, (y_1, \tilde{y})) = (t + y_1, \tilde{y}).$$

Hence, for any $t \in I$ and $y = (y_1, \tilde{y}) \in U$ such that $(t + y_1, \tilde{y}) \in U$, by (8.3) we have

$$\rho(\psi(t, y)) = \rho(t + y_1, \tilde{y}) = \phi(t + y_1, \rho(\tilde{y})) = \phi(t, \phi(y_1, \rho(\tilde{y}))) = \phi(t, \rho(y))$$

This is what we need to show. $\square$

We end the section by a proposition in analysis.

Proposition 8.4. *Suppose f is a C^1 vector field in an open set $D \subset \mathbb{R}^n$ and there is a closed nonempty ball $B \subset D$ such that f is nonzero on and nowhere tangent to the boundary of B. Then, f possesses a critical point in B.*

Proof. Let $\phi(t, x)$ be the local flow of f. Since, f is non-zero on and not tangent to the boundary of B, orbits at the boundary either flow into or out of B. We suppose they flow into B. In the other case, we can replace f by $-f$.

For $x \in B$, the solution $\phi(t, x)$ is defined and remains in B for all $t > 0$. Let $m > 0$ be a positive integer, and consider the mapping $x \to \phi_{\frac{1}{m}}(x)$. This is a continuous self-map of the closed ball B to itself. By the Brouwer fixed point theorem, it has a fixed point, say x_m. Since B is compact, the sequence $\{x_m\}$ has a subsequence x_{m_k} which converges, say to the point y as $k \to \infty$.

Let us show that $f(y) = 0$. If not, then by the flow box theorem, there are a neighborhood V of y in D and an interval $I_\epsilon = [-\epsilon, \epsilon]$ about 0 in $\mathbb{R}$ such that,

(i) for $z \in V$, the solution $\phi(t, z)$ is defined for all $t \in [-\epsilon, \epsilon]$;
(ii) $\phi(t_1, z) \neq \phi(t_2, z)$ for $t_1 \neq t_2 \in I_\epsilon$.

But, if k is large enough, then $x_{m_k} \in V$, and $\frac{1}{m_k} < \epsilon$. Then, $\phi_{\frac{1}{m}}(x_{m_k}) \neq x_{m_k}$ by (ii), which contradicts the definition of x_{m_k}. □