9. Two dimensional systems

We will apply some topology of the Euclidean plane to obtain information about two dimensional planar autonomous systems.

Definition 9.1. *Let* $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ *be the unit circle in the plane* $\mathbb{R}^2$. *A* Jordan curve *in the plane* $\mathbb{R}^2$ *(or a simple closed curve) in the plane* $\mathbb{R}^2$*) is the image of a 1-1 continuous map* $h : S^1 \to \mathbb{R}^2$.

Theorem 9.1 (Jordan Curve Theorem)**.** . *Let* γ *be a Jordan curve in the plane* $\mathbb{R}^2$. *Then,* $\mathbb{R}^2 \setminus \gamma$ *is the union of two disjoint open connected sets* S_1, S_2 *each of which have* γ *as boundary. Precisely one of the regions* S_1, S_2 *is bounded.*

Remark 9.2. *The book refers to the sets* S_i *as being arcwise connected. But open connected sets in the plane are arcwise connected.*

We will not prove this theorem here, instead referring to a course in topology. The bounded region of $\mathbb{R}^2 \setminus \gamma$ is frequently referred to as the *interior* of γ although it is *not* the interior in the sense of topology.

Unless otherwise stated, we will assume that f is a C^1 vector field defined in the plane $\mathbb{R}^2$ and, for each x, the solution $\phi(t, x)$ is defined for all $t \in \mathbb{R}$.

Let p be a regular point of the vector field f; i.e., $f(p) \neq 0$. Let L be a *closed tranversal* to f at p. This means that there is a C^1 diffeomorphism $h : [-1, 1] \to L$ such that $h(0) = p, h'(t) \neq 0$, and, for each $t \in [-1, 1], h'(t)$ is not a multiple of $f(h(t))$. Let L_0 be the interior of L; i.e., $L_0 = \{h(s) : -1 < s < 1\}$.

Let

$$V = \{q \in L_0 : \ \exists t_q > 0 \text{ s.t. } \phi(t_q, q) \in L_0, \phi(t, q) \notin L_0 \ \forall 0 < t < t_q\}.$$

The set V is the set of points in L_0 whose positive orbits return to L_0. Let $V' = \{\phi(t_q, q) : q \in V\}$. Set $W = h^{-1}V$, $W' = h^{-1}V'$ so that W and W' are subsets of $(-1, 1)$.

Define $g : W \to W'$ by $g(w) = h^{-1}\phi(t_{h(w)}, h(w))$.

Lemma 9.3. *The set* W *is open in* $(-1, 1)$ *and the function* g *is continuous on* W. *For any* $z \in W$ *for which the iterates* $z, g(z), \dots, g^n(z)$ *is defined, the sequence* $z, g(z), \dots, g^n(z)$ *is monotone in* $(-1, 1)$.

Proof. (W is open) It is enough to show that V is open in L. Fix $q \in V$, we show that there exists a neighborhood U of q in L_0 such that for any $z \in U$, $z \in V$ as well.

Take a flow-box B centered at $\phi(t_q, q)$ such that each connected component of an orbit in B is an arc which only meets L in one point. There is an $\epsilon_1 > 0$ such that if $u \in B$, then there is a $\eta(u) \in (-\epsilon_1, \epsilon_1)$ such that $\phi(\eta(u), u) \in L$. Moreover, in the flow-box coordinates on B, $\eta(u)$ is obviously a continuous function of u since $L \cap B$ is a curve transverse to the vector field. This implies that, in the standard coordinates on B, $\eta(u)$ is still a continuous function of u.

By continuity of $\phi(t_q, \cdot)$, there is a neighborhood U of q in L such that if $z \in U$, then $\phi(t_q, z) \in B$. Hence,

$$\phi(t_q + \eta(\phi(t_q, z)), z) = \phi(\eta(\phi(t_q, z)), \phi(t_q, z)) \in L.$$

So $z \in V$. This gives that V is open.

(g is continuous) The map $z \to \phi(t_z, z) = \phi(t_q + \eta(\phi(t_q, z)), z)$ is continuous because $\phi(t_q, \cdot)$, $\eta(\cdot)$ and $\phi(\cdot, \cdot)$ are all continuous. We get that g is continuous.

(g is Monotone) Consider a point $z \in W$. If $g(z) = z$ for all z, there is nothing to prove, so assume z is such that $g(z) > z$. In the opposite case in which $g(z) < z$ one proceeds similarly.

The solution curve from $h(z)$ to $h(g(z))$ together with the piece, say L_1 of L from $h(g(z))$ to $h(z)$ is a Jordan curve γ. Since solutions always cross L moving in the same direction, the forward orbit of a point in the interval $L_1 \setminus \{h(z)\}$ always lies in the same component of the complement of γ. If $g(g(z))$ is defined, then $g(g(z))$ must be greater than $g(z)$ since otherwise, the forward orbit of $h(g(z))$ would have to pass from one component of the complement of γ to the other one. Now the argument continues replacing z by $g(z)$. □

Denote by $\mathcal{O}_+(x) = \{\phi(t, x) : t \geq 0\}$ the forward orbit of x.

Corollary 9.4. *The ω-limit set $\omega(\gamma)$ of an orbit γ can intersect the interior L_0 of a transversal L in at most one point.*

So if $\gamma = \gamma(p)$ and $\omega(\gamma) \cap L_0 = \{p_0\}$, then either $\omega(\gamma) = \gamma$ and γ is a periodic orbit, or there exist $\{p_i\} \subset \mathcal{O}_+(x) \cap L_0$ such that p_i approaches p_0 monotonically.

Proof. The first part of the corollary is obvious. We only prove that second part.

We assume that L_0 is small enough that it fits inside a single flow box B.

Since $p_0 = \omega(p) \cap L_0$, there is a sequence $t_1 < t_2 < \ldots$ with $t_k \to \infty$ such that $\phi(t_k, p) \to p_0$ in L_0.

If for some $j < k$, $\phi(t_j, p) = \phi(t_k, p)$. Then, γ is periodic. It must be equal to its own $\omega-$limit set.

If for all $k < j$, $\phi(t_j, p) \neq \phi(t_k, p)$, then we can construct a Jordan curve using pieces of orbits $\phi(t_i, p), \phi(t_{i+1}, p)$ and pieces of L_0 as before shows that the forward orbit $\mathcal{O}_+(p)$ in L_0 approach p_0 monotonically in L_0 as required. $\square$

Corollary 9.5. *If some regular point of $\mathcal{O}_+(p)$ is also in $\omega(p)$, then $\mathcal{O}(p)$ is periodic.*

Proof. This is another corollary of the Jordan curve theorem and the flow box theorem. $\square$

Theorem 9.6. *bounded minimal set of a C^1 autonomous planar vector field is a critical point or a periodic orbit.*

Proof. If the minimal set is not a critical point, then it contains no critical points. But each of its orbits must be dense in the set. Thus, each of its points is in its own ω-limit sets. Hence, by the previous corollary, each of its orbits is periodic. Since it is minimal, it must be a single orbit. $\square$

Theorem 9.7 (Poincare-Bendixson Theorem)**.** *Suppose that $\mathcal{O}_+(x)$ is a bounded positive semi-orbit of an autonomous C^1 vector field f in the plane. If $\omega(x)$ does not contain a critical point, then $\omega(x)$ consists of a periodic orbit $\mathcal{O}(p)$. Either $\mathcal{O}(p) = \mathcal{O}(x)$ or $\mathcal{O}(p) = \overline{\mathcal{O}_+(x)} \setminus \mathcal{O}_+(x)$.*

Proof. Since $\mathcal{O}_+(x)$ is bounded, $\overline{\mathcal{O}_+(x)} \neq \emptyset$ and $\omega(x)$ is a non-empty invariant set. By hypothesis, it contains only regular points. It also contains a minimal set Σ which must be a periodic orbit, $\mathcal{O}(p)$.

Suppose $\mathcal{O}(p) \neq \mathcal{O}(x)$. So $\overline{\mathcal{O}(p)} \cap \mathcal{O}(x) = \emptyset$.

Since $\mathcal{O}(p) \subset \omega(x)$, $\mathcal{O}(p) \subset \overline{\mathcal{O}_+(x)} \setminus \mathcal{O}_+(x)$. Now we prove $\overline{\mathcal{O}_+(x)} \setminus \mathcal{O}_+(x) \subset \mathcal{O}(p)$.

Let L_0 be a small open transversal arc to the periodic orbit $\mathcal{O}(p)$ at p. Since p is in $\omega(x)$, there is a sequence $t_1 < t_2 < \dots$ with $t_i \to \infty$ such that $p_i := \phi(t_i, x) \in L_0$, and $p_i \to p$ as $i \to \infty$.

Let K_i be the region bounded by $\mathcal{O}(p)$, $\{\phi(t, x) : t_i \leq t \leq t_{i+1}\}$, and the subcurve from p_i to p_{i+1} in L_0. Since $\{\phi(t, x) : t \geq t_{i+1}\} \subset K_i$, $\overline{K}_i \supset \overline{\mathcal{O}_+(x)} \setminus \mathcal{O}_+(x)$. This is true for any $i \geq 0$, so $\cap_{i\geq 0}\overline{K}_i \supset \overline{\mathcal{O}_+(x)} \setminus \mathcal{O}_+(x)$. Since $\cap_{i\geq 0}\overline{K}_i = \mathcal{O}(p)$, we get what we need. $\square$

Corollary 9.8. *If $\phi(x)$ contains a periodic orbit $\mathcal{O}(p)$, then $\phi(x) = \mathcal{O}(p)$.*

Proof. Use the last part of proof of the theorem. $\square$

Lemma 9.9. *Suppose that $\omega(p)$ contains a regular point q, which is not in the orbit of p. Then, p is a wandering point and therefore p cannot be in $\omega(x)$ for any x.*

Proof. Take a small open transversal L_0 to q. Since $q \in \omega(p)$, there exists a sequence of points $\{p_i\} \subset \mathcal{O}(p) \cap L_0$ such that $p_i \to q$ monotonically as $i \to \infty$. Hence any p_i is a wandering point and so is any point in $\mathcal{O}(p)$.

Since $\omega(x)$ is contained in the nonwandering set for any x, the result follows. □

Theorem 9.10. *Suppose $\mathcal{O}_+(x)$ is a positive semi-orbit in a closed bounded subset K of the plane for a C^1 vector field f. Assume that K contains only a finite number of critical points. Then, one of the following holds.*

(i) *$\omega(x)$ is a critical point.*
(ii) *$\omega(x)$ is a periodic orbit.*
(iii) *$\omega(x)$ consists of a finite number of critical points and a set of orbits γ_i such that each γ_i has its ω-limit set $\omega(\gamma_i)$ and α-limit set $\alpha(\gamma_i)$ consisting of a critical point.*

Proof. If $\omega(x)$ contains no regular points, then since it is connected, it must consist of a single critical point. This is case (i).

Then we assume $\omega(x)$ contains at least one regular point, say p.

If $\mathcal{O}(p)$ is periodic, then by Corollary 9.8, which is case (ii).

If the orbit $O(p)$ is not periodic, then by Lemma 9.9 $\omega(p)$ cannot cantain any regular point since p is a nonwandering point. Therefore, $\omega(p)$ consists only of critical points. Since it is connected, it must be a single critical point. A similar argument works for $\alpha(p_0)$.

We have therefore proved that each regular non-periodic ω-limit point or α-limit point of x must have each of its own ω-limit and α-limit sets reducing to single critical points. □

Definition 9.2. *A* separatrix cycle *is a set consisting of finite number of critical points $p_1, p_2, \ldots, p_n$ and regular orbits $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that $\alpha(\gamma_i) = p_i$ and $\omega(\gamma_i) = p_{i+1}$ for each $1 \le i < n$, and $\alpha(\gamma_n) = p_n$ and $\omega(\gamma_n) = p_1$.*

A solution γ whose α and ω limit sets are critical points is called a separatrix.

Remark 9.11. *One way in which condition (iii) in Theorem 9.10 occures is that ω-limit set consists of separatrix cycles. There can also be several regular orbits whose α and ω limits are the same critical point.*

Definition 9.3. *A closed orbit γ is called a* limit cycle *if $\gamma = \alpha(x)$ or $\gamma = \omega(x)$ for some $x \notin \gamma$.*

Part of Hilbert 16th problem is *the determination of the upper bound for the number of limit cycles in two-dimensionial polynomial vector* *fields of degree n and an investigation of their relative positions*, and the problem remains unsolved.