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An Example of a Smooth Hyperbolic Measure with Countably Many Ergodic

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0.1. Introduction. We construct an example of a diffeomorphism with nonzero Lyapunov exponents with respect to a smooth invariant measure which has countably many ergodic components. More precisely we will prove the following result.

THEOREM 0.1. *There exists a C^∞ diffeomorphism f of the three dimensional torus $\mathbb{T}^3$ such that*

1. *f preserves the Riemannian volume μ on $\mathbb{T}^3$;*
2. *μ is a hyperbolic measure;*
3. *f has countably many ergodic components which are open* (mod 0).

0.2. Construction of the Diffeomorphism f. Let $A\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a linear hyperbolic automorphism. Passing if necessary to a power of A we may assume that A has at least two fixed points p and p'. Consider the map $F = A \times \mathrm{Id}$ of the three dimensional torus $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$. We will perturb F to obtain the desired map f.

Consider a countable collection of intervals $\{I_n\}_{n=1}^\infty$ on the circle $\mathbb{S}^1$, where

$$I_{2n} = [(n+2)^{-1}, (n+1)^{-1}], \quad I_{2n-1} = [1-(n+1)^{-1}, 1-(n+2)^{-1}].$$

Clearly, $\bigcup_{n=1}^\infty I_n = (0,1)$ and $\operatorname{int} I_n$ are pairwise disjoint.

By Proposition 0.2 below, for each n one can construct a C^∞ volume preserving ergodic diffeomorphism $f_n\colon \mathbb{T}^2 \times [0,1] \to \mathbb{T}^2 \times [0,1]$ which satisfies:

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1. $\|F - f_n\|_{C^n} \le e^{-n^2}$;
2. for all $0 \le m < \infty$, $D^m f_n|\mathbb{T}^2 \times \{z\} = D^m F|\mathbb{T}^2 \times \{z\}$ for $z = 0$ or 1;
3. f_n has nonzero Lyapunov exponents μ-almost everywhere.

Let $L_n \colon I_n \to [0,1]$ be the affine map and $\pi_n = (\mathrm{Id}, L_n) \colon \mathbb{T}^2 \times I_n \to \mathbb{T}^2 \times [0,1]$. We define the map f by setting $f|\mathbb{T}^2 \times I_n = \pi_n^{-1} \circ f_n \circ \pi_n$ for all n and $f|\mathbb{T}^2 \times \{0\} = F|\mathbb{T}^2 \times \{0\}$. Note that for every $n > 0$ and $0 \le m \le n$ we have

$$\begin{aligned}\|D^m F|\mathbb{T}^2 \times I_n - \pi_n^{-1} \circ D^m f_n \circ \pi_n\|_{C^n} &\le \|\pi_n^{-1} \circ (D^m F - D^m f_n) \circ \pi_n\|_{C^n}\\ &\le e^{-n^2} \cdot n^n \to 0\end{aligned}$$

as $n \to \infty$. It follows that f is C^∞ on M and has the required properties.

0.3. Main Proposition. The goal of this section is to prove the following statement. Set $I = [0,1]$.

Proposition 0.2. *For any $k \ge 2$ and $\delta > 0$, there exists a map g of the three dimensional manifold $M = \mathbb{T}^2 \times I$ such that:*

1. *g is a C^∞ volume preserving diffeomorphism of M;*
2. *$\|F - g\|_{C^k} \le \delta$;*
3. *for all $0 \le m < \infty$, $D^m g|\mathbb{T}^2 \times \{z\} = D^m F|\mathbb{T}^2 \times \{z\}$ for $z = 0$ and 1;*
4. *g is ergodic with respect to the Riemannian volume and has nonzero Lyapunov exponents almost everywhere.*

Before giving the formal proof let us outline the main idea. The result will be achieved in two steps. First applying a method of [**SW**] we construct a perturbation map which has nonzero *average* central exponent $\int_M \chi^c(x)\, d\mu(x) \ne 0$, where $\chi^c(x)$ denotes the Lyapunov exponent of x along the neutral subspace $E^c(x)$. We then further perturb this diffeomorphism using a method of [**NT**] to ensure that it has the accessibility property and therefore, is ergodic (see Section B.4 for details).

We believe that this approach works in a more general setting. Namely, we conjecture that the following statement holds.

Proposition 0.3. *Consider a one parameter family g_ε with $g_0 = F$. Then for sufficiently small ε, g_ε satisfies the conditions of Proposition 0.2 except for a positive codimension submanifold in the space of one parameter families.*

Proof. We proceed with the proof of Main Proposition. Consider the linear hyperbolic map A of the torus $\mathbb{T}^2$. We may assume that its eigenvalues are η and η^{-1}, where $\eta > 1$. Let p and p' be fixed points of A. Choose a number $\varepsilon_0 > 0$ such that $d(p, p') \ge 3\varepsilon_0$. Consider the local stable and unstable one-dimensional manifolds for A at points p and p' of "size" ε_0 and denote them respectively by $V^s(p)$, $V^u(p)$, $V^s(p')$, and $V^u(p')$.

Let us choose the smallest positive number n_1 such that the intersection $A^{-n_1}(V^s(p')) \cap V^u(p) \cap B(p, \varepsilon_0)$ consists of a single point which we denote by q_1 (here $B(p, \varepsilon_0)$ is the ball in $\mathbb{T}^2$ of radius ε_0 centered at p). Similarly, we choose the smallest positive number n_2 such that the intersection $A^{n_2}(V^u(p')) \cap V^s(p) \cap B(p, \varepsilon_0)$ consists of a single point which we denote by q_2.

Given a sufficiently small number $\varepsilon \in (0, \varepsilon_0)$,

$$\varepsilon \leq \frac{1}{2} \min\{d(p, q_1), d(p, q_2)\},$$

there is $\ell \geq 2$ such that

$$A^{-\ell}(q_1) \notin B(p, \varepsilon), \quad A^{-\ell-1}(q_1) \in B(p, \varepsilon). \tag{0.1}$$

We now choose $\varepsilon' \in (0, \varepsilon)$ such that $A^{-\ell-1}(q_1) \in B(p, \varepsilon')$.

Finally, we assume ε to be so small that for some $q \in \mathbb{T}^2$ we have

$$B(p, \varepsilon) \cap (A^{-n_1}(V^s(p')) \cup A^{n_2}(V^u(p'))) = \varnothing,$$

$$A^i(B(q, \varepsilon)) \cap B(q, \varepsilon) = \varnothing, \quad A^i(B(q, \varepsilon)) \cap B(p, \varepsilon) = \varnothing$$

for $i = 1, \ldots, N$, where $N > 0$ will be determined later, and $\varepsilon = \varepsilon(N)$.

Set $\Omega_1 = B(p, \varepsilon_0) \times I$ and $\Omega_2 = B^{uc}(\bar{q}, \varepsilon_0) \times B^s(\bar{q}, \varepsilon_0)$, where $\bar{q} = (q, 1/2)$ and $B^{uc}(\bar{q}, \varepsilon_0) \subset V^u(q) \times I$ and $B^s(\bar{q}, \varepsilon_0) \subset V^s(q)$ are balls of radius ε_0 about $\bar{q}$.

After this preliminary considerations we describe the construction of the map g.

Consider the coordinate system in Ω_1 originated at $(p, 0) \in M$ with x, y, and z-axes to be unstable, stable, and neutral directions respectively for the map F. If a point $w = (x, y, z) \in \Omega_1$ and $F(w) \in \Omega_1$ then $F(w) = (\eta x, \eta^{-1} y, z)$.

Choose a C^∞ function $\xi \colon I \to \mathbb{R}^+$ satisfying:

1. $\xi(z) > 0$ on $(0, 1)$;
2. $\xi^{(i)}(0) = \xi^{(i)}(1) = 0$ for $i = 0, 1, \ldots, k$;
3. $\|\xi\|_{C^k} \leq \delta$.

We also choose two C^∞ functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ which are defined on the interval $(-\varepsilon_0, \varepsilon_0)$ and satisfy

4. $\varphi(x) = \varphi_0$ if $x \in (-\varepsilon', \varepsilon')$ and $\psi(y) = \psi_0$ if $y \in (-\varepsilon', \varepsilon')$, where φ_0 and ψ_0 are positive constants;
5. $\varphi(x) = 0$ if $|x| \geq \varepsilon$; $\psi(y) \geq 0$ for any y and $\psi(y) = 0$ if $|y| \geq \varepsilon$;
6. $\|\varphi\|_{C^k} \leq \delta$, $\|\psi\|_{C^k} \leq \delta$;
7. $\int_0^{\pm\varepsilon} \varphi(s)\, ds = 0$.

We now define the vector field X on Ω_1 by

$$X(x, y, z) = \left(-\psi(y)\xi'(z) \int_0^x \varphi(s) ds, 0, \psi(y)\xi(z)\varphi(x) \right).$$

It is easy to check that X is a divergence free vector field supported on $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times I$.

We define the map h_t on Ω_1 to be the time t map of the flow generated by X and we set $h_t = \mathrm{Id}$ on the complement of Ω_1. It is easy to see that

h_t is a C^∞ volume preserving diffeomorphism of M which preserves the y coordinate (the stable direction for the map F).

Consider now the coordinate system in Ω_2 originated at $(q, 1/2)$ with x, y, and z-axes to be unstable, stable, and neutral directions respectively. We then switch to the cylindrical coordinate system (r, θ, y), where $x = r\cos\theta$, $y = y$, and $z = r\sin\theta$.

Consider a C^∞ function $\rho\colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^+$ satisfying:

8. $\rho(r) > 0$ if $0.2\varepsilon' \le r \le 0.9\varepsilon$ and $\rho(r) = 0$ if $r \le 0.1\varepsilon'$ or $r \ge \varepsilon$;
9. $\|\rho\|_{C^k} \le \delta$.

We define now the map $\widetilde{h}_\tau$ on Ω_2 by

$$\widetilde{h}_\tau(r, \theta, y) = (r, \theta + \tau\psi(y)\rho(r), y). \tag{0.2}$$

and we set $\widetilde{h}_\tau = \mathrm{Id}$ on $M \setminus \Omega_2$. It is easy to see that for every τ the map $\widetilde{h}_\tau$ is a C^∞ volume preserving diffeomorphism of M.

Let us set $g = g_{t\tau} = h_t \circ F \circ \widetilde{h}_\tau$. For all sufficiently small $t > 0$ and τ, the map $g_{t\tau}$ is C^k close to F and hence, is a partially hyperbolic (in the narrow sense) C^∞ diffeomorphism of M. It preserves the Riemannian volume in M and is ergodic by Lemma 0.4. It remains to show that $g_{t\tau}$ has nonzero Lyapunov exponents almost everywhere.

Denote by $E^s_{t\tau}(w)$, $E^u_{t\tau}(w)$, and $E^c_{t\tau}(w)$ the stable, unstable, and neutral subspaces at a point $w \in M$ for the map $g_{t\tau}$. It suffices to show that for almost everywhere point $w \in M$ and every vector $v \in E^c_\tau(w)$, the Lyapunov exponent $\chi(w, v) \neq 0$.

Set $\kappa_{t\tau}(w) = Dg_{t\tau}|E^u_{t\tau}(w)$, $w \in M$. By Lemma 0.7, for all sufficiently small $\tau > 0$,

$$\int_M \log \kappa_{0\tau}(w)\, dw < \log \eta.$$

The subspace $E^u_{t\tau}(w)$ depends continuously on t and τ (for a fixed w; for details see the paper by Burns, Pugh, Shub, and Wilkinson in this volume) and hence, so does $\kappa_{t\tau}$. It follows that for all sufficiently small $\tau > 0$, there is $t > 0$ such that

$$\int_M \log \kappa_{t\tau}(w)\, dw < \log \eta.$$

Denote by $\chi^s_{t\tau}(w)$, $\chi^u_{t\tau}(w)$, and $\chi^c_{t\tau}(w)$ the Lyapunov exponents of $g_{t\tau}$ at the point $w \in M$ in the stable, unstable, and neutral directions respectively (since these directions are on-dimensional the Lyapunov exponents do not depend on the vector). By the ergodicity of $g_{t\tau}$, we have that for almost every $w \in M$,

$$\chi^u_{t\tau}(w) = \lim_{n\to\infty} \frac{1}{n} \log \prod_{i=0}^{n-1} \kappa_{t\tau}(g^i_{t\tau}(w)).$$

By the Birkhoff ergodic theorem, we get

$$\chi^u_{t\tau}(w) = \int_M \log \kappa_{t\tau}(w)\, dw < \log \eta.$$

Since $E^s_{t\tau}(w) = E^s_{00}(w) = E^s_F(w)$ for every t and τ, we conclude that $\chi^s_{t\tau}(w) = -\log\eta$ for almost every $w \in M$. Since $g_{t\tau}$ is volume preserving,

$$\chi^s_{t\tau}(w) + \chi^u_{t\tau}(w) + \chi^c_{t\tau}(w) = 0$$

for almost every $w \in M$. It follows that $\chi^c_{t\tau}(w) \neq 0$ for almost every $w \in M$ and hence, $g_{t\tau}$ has nonzero Lyapunov exponents almost everywhere. This completes the proof of the proposition. □

0.4. Ergodicity of the Map $g_{t\tau}$.

LEMMA 0.4. *For every sufficiently small $t > 0$ and $\tau \geq 0$ the map $g_{t\tau}$ is ergodic.*

PROOF. Consider a partially hyperbolic (in the narrow sense) diffeomorphism f of a compact Riemannian manifold M preserving the Riemannian volume. Two points x, $y \in M$ are called *accessible (with respect to f* if they can be joined by a piecewise differentiable piecewise nonsingular path which consists of segments tangent to either E^u or E^s. The diffeomorphism f satisfies the *essential accessibility property* if almost any two points in M (with respect to the Riemannian volume) are accessible. We will show that the map $g_{t\tau}$ has the essential accessibility property. The ergodicity of the map will then follow from the result by Pugh and Shub (see [**PS**]; see also the paper by Burns, Pugh, Shub, and Wilkinson in this volume).

Given a point $w \in M$, denote by $\mathcal{A}(w)$ the set of points $q \in M$ such that w and q are accessible. Set $I_p = \{p\} \times (0, 1)$.

LEMMA 0.5. *For every $z \in (0, 1)$,*

$$\mathcal{A}(p, z) \supset I_p. \tag{0.3}$$

PROOF OF LEMMA 0.5. We use the coordinate system (x, y, z) in Ω_1 described above. Since the map h_t preserves the center leaf I_p, we have that

$$h_t(0, 0, z) = (h^1_t(0, 0, z), h^2_t(0, 0, z), h^3_t(0, 0, z)) = (0, 0, h^3_t(0, 0, z))$$

for $z \in (0, 1)$. It suffices to show that for every $z \in (0, 1)$,

$$\mathcal{A}(p, z) \supset \{(p, a) : a \in [(h^{-\ell}_t)^3(p, z), z]\}, \tag{0.4}$$

where ℓ is chosen by (0.1). In fact, since accessibility is a transitive relation and $h^{-n}_t(p, z) \to (p, 0)$ for any $z \in (0, 1)$, (0.4) implies that $\mathcal{A}(p, z) \supset \{(p, a) : a \in (0, z]\}$. Since this holds true for all $z \in (0, 1)$ and accessibility is a reflexive relation, we obtain (0.3).

Now we proceed with the proof of (0.4).

Let $q_1 \in V^u_{t\tau}(p)$ and $q_2 \in V^s_{t\tau}(p)$ be two points constructed in Section 0.3. The intersection $V^s_{t\tau}(q_1) \cap V^u_{t\tau}(q_2)$ is not empty and consists of a single point q_3. We will prove that for any $z_0 \in (0, 1)$, there exist $z_i \in (0, 1)$, $i = 1, 2, 3, 4$ such that

$$\begin{aligned} &(q_1, z_1) \in V^u_{t\tau}((p, z_0)), \quad (q_3, z_3) \in V^s_{t\tau}((q_1, z_1)), \\ &(q_2, z_2) \in V^u_{t\tau}((q_3, z_3)), \quad (p, z_4) \in V^s_{t\tau}((q_2, z_2)) \end{aligned}$$

and

$$z_4 \le (h_t^{-\ell})^3(p, z_0). \tag{0.5}$$

This means that $(p, z_4) \in \mathcal{A}(p, z_0)$. By continuity, we conclude that

$$\{(p, a) : a \in [z_4, z_0]\} \subset \mathcal{A}(p, z_0)$$

and (0.4) follows.

Since $g_{t\tau}$ preserves the xz-plane, we have that $V^{uc}_{t\tau}((p, z_0)) = V^{uc}_F((p, z_0))$. Hence, there is a unique $z_1 \in (0, 1)$ such that $(q_1, z_1) \in V^u_{t\tau}((p, z_0))$. Notice that

$$g^{-n}_{t\tau}(p, z_0) = (p, h_t^{-n}((p, z_0)), \quad g^{-n}_{t\tau}(q_1, z_1) = (A^{-n}q_1, z_1)$$

for $n \le \ell$. This is true because the points $A^{-n}q_1$, $n = 0, 1, \ldots, \ell$ lie outside the ε-neighborhood of I_p, where the perturbation map $h_t = \mathrm{Id}$. Similarly, since the points $A^{-n}q_1$, $n > \ell$ lie inside the ε'-neighborhood of I_p, and the third component of h_t depends only on the z-coordinate, we have

$$g^{-n}_{t\tau}(q_1, z_1) = (A^{-n}q_1, h_t^{-n+\ell}z_1).$$

Since $d(g^{-n}_{t\tau}((p, z_0)), g^{-n}_{t\tau}((q_1, z_1))) \to 0$ as $n \to \infty$, we have

$$d(h_t^{-n}((p, z_0)), h_t^{-n+\ell}((p, z_1))) \to 0$$

as $n \to \infty$. It follows that $z_1 = (h_t^{-\ell})^3((p, z_0))$.

By the construction of the map h_t (that is $h_t = \mathrm{Id}$ outside Ω_1) the sets $A^{-n_1}V^s_{t\tau}(p')$ and $A^{n_2}V^u_{t\tau}(p')$ are pieces of horizontal lines. This means that $z_2 = z_3 = z_1$.

Since the third component of h_t is nondecreasing from (q_2, z_2) to (p, z_4) along $V^s_{t\tau}(p)$, we conclude that $z_4 \le z_3 = z_1 = (h_t^{-\ell})^3(p, z_0)$ and thus (0.5) holds. □

The essential accessibility property follows from Lemma 0.5 and the following statement.

Lemma 0.6 (see [**NT**]). *Assume that any two points in I_p are accessible. Then the map $g_{t\tau}$ satisfies the essential accessibility property.*

Proof of Lemma 0.6. It is easy to see that for any two points $x, y \in M$ which do not lie on the boundary of M one can find points $x', y' \in I_p$ such that the pairs (x, x') and (y, y') are accessible. By Lemma 0.5 the points x', y' are accessible. Since accessibility is a transitive relation the result follows. □

□

0.5. Hyperbolicity of the Map $g_{0\tau}$. In this section we show that for all sufficiently small τ, the map $g_{0\tau}$ has nonzero average Lyapunov exponent in the central direction. Since this map is ergodic this implies that $g_{0\tau}$ has nonzero Lyapunov exponents almost everywhere.

LEMMA 0.7. *For any sufficiently small $\tau > 0$,*

$$\int_M \log \kappa_{0\tau}(w)dw < \log \eta. \tag{0.6}$$

PROOF. Our approach is an elaboration of an argument in [**SW**].

For any $w \in M$, we introduce the coordinate system in T_wM associated with the splitting $E^u_F(w) \oplus E^s_F(w) \oplus E^c_F(w)$. Given $\tau \geq 0$ and $w \in M$, there exists a unique number $\alpha_\tau(w)$ such that the vector $v_\tau(w) = (1, 0, \alpha_\tau(w))^t$ lies in $E^u_{0\tau}(w)$ (where t denotes the transpose). Since the map $\widetilde{h}_\tau$ preserves the y coordinate, by the definition of the function $\alpha_\tau(w)$, one can write the vector $Dg_{0\tau}(w)v_\tau(w)$ in the form

$$Dg_{0\tau}(w)v_\tau(w) = (\bar{\kappa}_\tau(w), 0, \bar{\kappa}_\tau(w)\alpha_\tau(g_{t0}(w)))^t \tag{0.7}$$

for some $\bar{\kappa}_\tau(w) > 1$. Since the expanding rate of $Dg_{0\tau}(w)$ along its unstable direction is $\kappa_{0\tau}(w)$ we obtain that

$$\kappa_{0\tau}(w) = \bar{\kappa}_\tau(w)\frac{\sqrt{1 + \alpha_\tau(g_{0\tau}(w))^2}}{\sqrt{1 + \alpha_\tau(w)^2}}.$$

Since $E^u_{0\tau}(w)$ is close to $E^u_{00}(w)$ the function $\alpha_\tau(w)$ is uniformly bounded. Using the fact that the map $g_{0\tau}$ preserves the Riemannian volume we find that

$$L_\tau = \int_M \log \kappa_{0\tau}(w)\, dw = \int_M \log \bar{\kappa}_\tau(w)\, dw. \tag{0.8}$$

Consider the map $\widetilde{h}_\tau$. Since it preserves the y-coordinate using (0.2), we can write that

$$\widetilde{h}_\tau(x, y, z) = (r \cos \sigma, y, r \sin \sigma),$$

where $\sigma = \sigma(\tau, r, \theta, y) = \theta + \tau\psi(y)\rho(r)$. Therefore, the differential

$$D\widetilde{h}_\tau \colon E^u_F(w) \oplus E^c_F(w) \to E^u_F(g_{0\tau}(w)) \oplus E^c_F(g_{0\tau}(w))$$

can be written in the matrix form

$$\begin{aligned} D\widetilde{h}_\tau(w) &= \begin{pmatrix} A(\tau, w) & B(\tau, w) \\ C(\tau, w) & D(\tau, w) \end{pmatrix} \\ &= \begin{pmatrix} r_x \cos \sigma - r\sigma_x \sin \sigma & r_y \cos \sigma - r\sigma_y \sin \sigma \\ r_x \sin \sigma + r\sigma_x \cos \sigma & r_y \sin \sigma + r\sigma_y \cos \sigma \end{pmatrix}, \end{aligned}$$

where

$$r_x = \frac{\partial r}{\partial x} = \frac{x}{r} = \cos \theta, \quad r_z = \frac{\partial r}{\partial z} = \frac{y}{r} = \sin \theta,$$

$$\sigma_x = \frac{\partial \sigma}{\partial x} = \frac{-z}{r^2} + \frac{z}{r}\tau\widetilde{\rho}_r(y, r) = \frac{\sin \theta}{r} + \tau\widetilde{\rho}_r(y, r) \cos \theta,$$

$$\sigma_z = \frac{\partial \sigma}{\partial z} = \frac{x}{r^2} + \frac{x}{r}\tau\widetilde{\rho}_r(y, r) = \frac{\cos \theta}{r} + \tau\widetilde{\rho}_r(y, r) \sin \theta,$$

and $\widetilde{\rho}(y,r)=\psi(y)\rho(r)$. It is easy to check that

$$\begin{aligned}
A&=A(\tau,w)=1-\tau r\widetilde{\rho}_r\sin\theta\cos\theta-\frac{\tau^2\widetilde{\rho}^2}{2}-\tau^2r\widetilde{\rho}\widetilde{\rho}_r\cos^2\theta+O(\tau^3),\\
B&=B(\tau,w)=-\tau\widetilde{\rho}-\tau r\widetilde{\rho}_r\sin^2\theta-\tau^2r\widetilde{\rho}\widetilde{\rho}_r\sin\theta\cos\theta+O(\tau^3),\\
C&=C(\tau,w)=\tau\widetilde{\rho}+\tau r\widetilde{\rho}_r\cos^2\theta-\tau^2r\widetilde{\rho}\widetilde{\rho}_r\sin\theta\cos\theta+O(\tau^3),\\
D&=D(\tau,w)=1+\tau r\widetilde{\rho}_r\sin\theta\cos\theta-\frac{\tau^2\widetilde{\rho}^2}{2}-\tau^2r\widetilde{\rho}\widetilde{\rho}_r\sin^2\theta+O(\tau^3).
\end{aligned}\tag{0.9}$$

By Lemma 0.8 below, we have

$$L_\tau=\log\eta-\int_M\log(D(\tau,w)-\eta B(\tau,w)\alpha_\tau(g_{0\tau}(w)))dw.$$

By Lemma 0.9, we have

$$\frac{dL_\tau}{d\tau}\Big|_{\tau=0}=0,\qquad \frac{d^2L_\tau}{d\tau^2}\Big|_{\tau=0}<0.$$

So we can choose τ so small that $L_\tau\neq\log\eta$. $\square$

Lemma 0.8.

$$L_\tau=\log\eta-\int_M\log(D(\tau,w)-\eta B(\tau,w)\alpha_\tau(g_{0\tau}(w)))dw.$$

Proof of Lemma 0.8. Since $g_{0\tau}=h_0\circ F\circ\widetilde{h}_\tau=F\circ\widetilde{h}_\tau$, we have that

$$D_\tau(w)=Dg_{0\tau}(w)|E^u_{0\tau}(w)\oplus E^c_{0\tau}(w)=\begin{pmatrix}\eta A(\tau,w)&\eta B(\tau,w)\\C(\tau,w)&D(\tau,w)\end{pmatrix}.$$

By (0.7),

$$D_\tau(w)\begin{pmatrix}1\\\alpha_\tau(w)\end{pmatrix}=\begin{pmatrix}\eta A(\tau,w)+\eta B(\tau,w)\,\alpha_\tau(w)\\C(\tau,w)+D(\tau,w)\,\alpha_\tau(w)\end{pmatrix}=\begin{pmatrix}\kappa_\tau(w)\\\kappa_\tau(w)\,\alpha_\tau(g_{0\tau}(w))\end{pmatrix}.\tag{0.10}$$

Since $\widetilde{h}_\tau$ is volume preserving, $AD-BC=1$ and therefore,

$$A+B\alpha=\frac{1}{D}+\frac{B}{D}(C+D\alpha).$$

Comparing the components in (0.10), we obtain

$$\begin{aligned}
\kappa_\tau(w)=&\eta(A(\tau,w)+B(\tau,w)\,\alpha_\tau(w))\\
=&\eta\left(\frac{1}{D(\tau,w)}+\frac{B(\tau,w)}{D(\tau,w)}(C(\tau,w)+D(\tau,w)\,\alpha_\tau(w))\right)\\
=&\eta\left(\frac{1}{D(\tau,w)}+\frac{B(\tau,w)}{D(\tau,w)}(\kappa_\tau(w)\,\alpha_\tau(g_{0\tau}(w)))\right).
\end{aligned}$$

Solving for $\kappa_\tau(w)$, we get

$$\kappa_\tau(w)=\frac{\eta}{D(\tau,w)-\eta B(\tau,w)\,\alpha_\tau(g_{0\tau}(w))}.$$

The desired result follows from (0.8). $\square$

LEMMA 0.9.
$$\frac{dL_\tau}{d\tau}\Big|_{\tau=0} = 0, \qquad \frac{d^2L_\tau}{d\tau^2}\Big|_{\tau=0} < 0. \tag{0.11}$$

PROOF OF LEMMA 0.9. In order to simplify notations we set $D'_\tau = \frac{\partial D}{\partial\tau}$, $B'_\tau = \frac{\partial B}{\partial\tau}$, $C'_\tau = \frac{\partial C}{\partial\tau}$, $D''_{\tau\tau} = \frac{\partial^2 D}{\partial\tau^2}$, and $B''_{\tau\tau} = \frac{\partial^2 B}{\partial\tau^2}$. Since the function $\alpha_\tau(w)$ is differentiable over τ (see the paper by Burns, Pugh, Shub, and Wilkinson in this volume) by Lemma 0.8, we find

$$\frac{dL_\tau}{d\tau} = -\int_M \frac{D'_\tau - \eta B'_\tau\alpha(g_{0\tau}(w)) - \eta B\frac{\partial\alpha_\tau(w)}{\partial\tau}(g_{0\tau}(w))}{D(\tau,w) - \eta B(\tau,w)\alpha_\tau(w)(g_{0\tau}(w))}\, dw$$

and therefore,

$$\begin{aligned}\frac{d^2L_\tau}{d\tau^2} = &\int_M \left(\frac{D'_\tau - \eta B'_\tau\alpha(g_{0\tau}(w)) - \eta B(\tau,w)\frac{\partial\alpha_\tau(w)}{\partial\tau}(g_{0\tau}(w))}{D(\tau,w) - \eta B(\tau,w)\alpha_s(g_{0\tau}(w))}\right)^2 dw\\ &-\int_M \frac{E(\tau,w)}{D(\tau,w) - \eta B(\tau,w)\,\alpha_\tau(g_{0\tau}(w))}dw,\end{aligned}$$

where

$$\begin{aligned}E(\tau,w) =&D''_{\tau\tau} - \eta B''_{\tau\tau}\alpha(g_{0\tau}(w))\\ &- \eta B(\tau,w)\frac{\partial^2\alpha_\tau(w)}{\partial\tau^2}(g_{0\tau}(w)) - 2\eta B'_\tau\frac{\partial\alpha_\tau(w)}{\partial\tau}(g_{0\tau}(w)).\end{aligned}$$

Note that for all $w \notin \Omega_2$,

$$A(\tau,w) = D(\tau,w) = 1, \quad C(\tau,w) = B(\tau,w) = 0$$

and for all $w \in M$,

$$A(0,w) = D(0,w) = 1, \quad C(0,w) = B(0,w) = 0, \quad \alpha_0(w) = 0.$$

It follows that

$$\frac{dL_\tau}{d\tau}\Big|_{\tau=0} = \int_{\Omega_2} D'_\tau\, dw, \tag{0.12}$$

and also that

$$\frac{d^2L_\tau}{d\tau^2}\Big|_{\tau=0} = \int_{\Omega_2}\left[(D'_\tau)^2 - D''_{\tau\tau} + 2\eta B'_\tau\frac{\partial\alpha_\tau(w)}{\partial\tau}(g_{0\tau}(w))\right]_{\tau=0} dw. \tag{0.13}$$

By (0.9), we obtain that

$$D'_\tau(0,w) = r\widetilde{\rho}_r(r)\sin\theta\cos\theta$$

and hence,

$$\int_{\Omega_2} D'_\tau dw = 0.$$

Therefore, (0.12) implies the equality in (0.11).

We now proceed with the inequality in (0.11). Applying Lemma 0.10 below we obtain that

$$\frac{\partial\alpha}{\partial\tau}(g_{0\tau}(w))\Big|_{\tau=0} = \frac{C'_\tau(0,w)}{\eta} + \sum_{n=1}^{\infty}\frac{C'_\tau(0,g_{00}^{-n}(w))}{\eta^{n+1}}.$$

It follows that

$$2\eta B'_\tau(0,w)\frac{\partial\alpha}{\partial\tau}(g_{0\tau}(w))\Big|_{\tau=0} =2B'_\tau(0,w)C'_\tau(0,w) + 2B'_\tau(0,w)\sum_{n=1}^{\infty}\frac{C'_\tau(0,g_{00}^{-n}(w))}{\eta^n}.$$

First, we evaluate the term

$$\mathcal{F}(w) = D'_\tau(0,w)^2 - D''_{\tau\tau}(0,w) + 2B'_\tau(0,w)C'_\tau(0,w).$$

Using (0.9), we find that

$$\begin{aligned}\mathcal{F}(w) =&(r\widetilde{\rho}_r\sin\theta\cos\theta)^2 + (\widetilde{\rho}^2 + 2r\widetilde{\rho}\widetilde{\rho}_r\sin^2\theta)\\ &- 2(\widetilde{\rho}+r\widetilde{\rho}_r\sin^2\theta)(\widetilde{\rho}+r\widetilde{\rho}_r\cos^2\theta)\\ =&-\widetilde{\rho}^2-(r\widetilde{\rho}_r\sin\theta\cos\theta)^2-2r\widetilde{\rho}\widetilde{\rho}_r\cos^2\theta.\end{aligned} \tag{0.14}$$

Recall that $\Omega_2 = B^{uc}(\bar{q},\varepsilon_0)\times B^s(\bar{q},\varepsilon_0)$ and $\widetilde{\rho}(r)=0$ if $r\geq\varepsilon$. We have

$$\int_{\Omega_2} 2r\widetilde{\rho}\widetilde{\rho}_r\cos^2\theta\,dw = \int_{-\varepsilon_0}^{\varepsilon_0}dy\int_0^{2\pi}2\cos^2\theta\,d\theta\int_0^{\varepsilon}r^2\widetilde{\rho}\widetilde{\rho}_r\,dr. \tag{0.15}$$

Since $0=\widetilde{\rho}(0)=\widetilde{\rho}(\varepsilon)$ (by the definition of the function ρ), we find that

$$\int_0^{\varepsilon}r^2\widetilde{\rho}\widetilde{\rho}_r\,dr = \frac{1}{2}r^2\widetilde{\rho}^2\Big|_0^{\varepsilon} - \int_0^{\varepsilon}r\widetilde{\rho}^2\,dr = -\int_0^{\varepsilon}r\widetilde{\rho}^2\,dr. \tag{0.16}$$

We also have that

$$\int_0^{2\pi}2\cos^2\theta\,d\theta = \int_0^{2\pi}d\theta. \tag{0.17}$$

It follows from (0.15)–(0.17) that

$$-\int_{\Omega_2}2r\widetilde{\rho}\widetilde{\rho}_r\cos^2\theta\,dw = \int_{\Omega_2}r\widetilde{\rho}^2\,dw \leq \varepsilon\int_{\Omega_2}\widetilde{\rho}^2\,dw. \tag{0.18}$$

Arguing similarly one can show that

$$-\int_{\Omega_2}r\widetilde{\rho}_r\sin\theta\cos\theta\,dw = -\frac{1}{8}\int_{\Omega_2}(r\widetilde{\rho})^2\,dw \tag{0.19}$$

Thus we conclude using (0.14), (0.18), and (0.19) that

$$\int_{\Omega_2}\mathcal{F}(0,w)\,dw \leq -(1-\varepsilon)\int_{\Omega_2}\widetilde{\rho}^2\,dw - \frac{1}{8}\int_{\Omega_2}(r\widetilde{\rho})^2\,dw < 0. \tag{0.20}$$

We now evaluate the remaining term

$$\mathcal{G}(0,w) = \sum_{n=1}^{\infty}\frac{1}{\eta^i}\int_{\Omega_2}2B'_\tau(0,w)C'_\tau(0,g_{00}^{-n}(w))\,dw.$$

Since the map $g_{00} = F$ preserves the Riemannian volume we obtain that

$$\begin{aligned}\int_{\Omega_2} 2B'_\tau(0,w)C'_\tau(0,g_{00}^{-n}(w))\,dw &\le \int_{\Omega_2} B'_\tau(0,w)^2\,dw + \int_{\Omega_2} C'_\tau(0,g_{00}^{-n}(w))^2\,dw\\ &= \int_{\Omega_2} B'_\tau(0,w)^2\,dw + \int_{\Omega_2} C'_\tau(0,w)^2\,dw\end{aligned}$$

Applying (0.9), we find that

$$\begin{aligned}&\int_{\Omega_2} B'_\tau(0,w)^2\,dw + \int_{\Omega_2} C'_\tau(0,w)^2\,dw\\ &= \int_{\Omega_2} (\widetilde{\rho} + r\widetilde{\rho}_r \sin^2\theta)^2\,dw + \int_{\Omega_2} (\widetilde{\rho} + r\widetilde{\rho}_r \cos^2\theta)^2\,dw\\ &\le 4\left(\int_{\Omega_2} \widetilde{\rho}^2\,dw + \int_{\Omega_2} r^2\widetilde{\rho}_r^2\,dw\right).\end{aligned}$$

It follows that for sufficiently large $N > 0$ (which does not depend on ε)

$$\sum_{i=N}^{\infty} \frac{1}{\eta^i} \int_{\Omega_2} 2B'_\tau(0,w)C'_\tau(0,g_{00}^{-i}(w))\,dw \le \frac{1}{10}\left(\int_{\Omega_2} \widetilde{\rho}^2\,dw + \int_{\Omega_2} r^2\widetilde{\rho}_r^2\,dw\right). \tag{0.21}$$

Note that if $g_{00}^{-n}\Omega_2 \cap \Omega_2 = \varnothing$, then $B'_\tau(0,w)C'_\tau(0,g_{00}^{-n}(w)) = 0$ for all w. Hence,

$$\int_{\Omega_2} 2B'_\tau(0,w)C'_\tau(0,g_{00}^{-n}(w))\,dw = 0.$$

We may choose the point q and a small ε such that $g_{00}^{-n}\Omega_2 \cap \Omega_2 = F^{-n}\Omega_2 \cap \Omega_2 = \varnothing$ for all $n = 1, 2, \dots, N$. It follows from (0.13), (0.20), and (0.21) that

$$\begin{aligned}\frac{d^2 L_\tau}{d\tau^2}\Big|_{\tau=0} &= \int_{\Omega_2} \mathcal{F}(0,w)\,dw + \int_{\Omega_2} \mathcal{G}(0,w)\,dw\\ &\le -\left(\frac{9}{10} - \varepsilon\right)\int_{\Omega_2} \widetilde{\rho}^2\,dw - \frac{1}{40}\int_{\Omega_2} r^2\widetilde{\rho}_r^2\,dw < 0.\end{aligned}$$

The desired result follows. ☐

LEMMA 0.10.

$$\frac{\partial\alpha}{\partial\tau}(g_{0\tau}(w))\Big|_{\tau=0} = \sum_{n=0}^{\infty} \frac{C'_\tau(0,g_{00}^{-n}(w))}{\eta^{n+1}}.$$

PROOF OF LEMMA 0.10. Define

$$R(\tau,w,\alpha) = \frac{C(\tau,w) + D(\tau,w)\alpha}{\eta(A(\tau,w) + B(\tau,w)\alpha)}.$$

It follows from (0.7) that

$$\alpha_\tau(g_{0\tau}(w)) = R(\tau,w,\alpha_\tau(w)). \tag{0.22}$$

By (0.7) and (0.9), we have

$$\frac{\partial R}{\partial \tau}\Big|_{\tau=0} = \frac{(C'_\tau + D'_\tau\alpha)(A+B\alpha)+(C+D\alpha)(A'_\tau+B'_\tau\alpha)}{\eta(A+B\alpha)^2}\Big|_{\tau=0} = \frac{C'_\tau(0,w)}{\eta}.$$

Since $A(0,w)$, $B(0,w)$, $C(0,w)$, and $D(0,w)$ are constant functions over $w=(x,y,z)$ we obtain that

$$\frac{\partial H}{\partial x}\Big|_{\tau=0} = \frac{\partial H}{\partial z}\Big|_{\tau=0} = 0$$

for $H = A$, B, C, D. This implies that

$$\frac{\partial R}{\partial x}\Big|_{\tau=0} = \frac{\partial R}{\partial z}\Big|_{\tau=0} = 0.$$

Since $AD-BC=1$,

$$\frac{\partial R}{\partial \alpha}\Big|_{\tau=0} = \frac{AD-BC}{\eta(A+B\alpha)^2}\Big|_{\tau=0} = \frac{1}{\eta}.$$

It follows from (0.22) that

$$\frac{\partial \alpha}{\partial \tau}(g_{0\tau}(w))\Big|_{\tau=0} = \frac{C'_\tau(0,w)}{\eta} + \frac{1}{\eta}\cdot\frac{\partial \alpha}{\partial \tau}(w)\Big|_{\tau=0}.$$

Since this inequality holds for any w, replacing w with $g_{0\tau}^{-1}(w)$ we obtain

$$\frac{\partial \alpha}{\partial \tau}(w)\Big|_{\tau=0} = \frac{C'_\tau(0,g_{0\tau}^{-1}(w))}{\eta} + \frac{1}{\eta}\cdot\frac{\partial \alpha}{\partial \tau}(g_{0\tau}^{-1}(w))\Big|_{\tau=0}.$$

The result follows by induction. □

References

[NT] V. Niţică and A. Török, *An open and dense set of stably ergodic diffeomorphisms in a neighborhood of a nonergodic one*, Topology, to appear.

[PS] C. Pugh and M. Shub, *Stable ergodicity and julienne quasi-conformality*, J. Eur. Math. Soc. **2** (2000), no. 1, 1–52.

[SW] M. Shub and A. Wilkinson, *Pathological foliations and removable zero exponents*, Invent. Math. **139** (2000), 495–508.

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