ERGODIC COMPONENTS OF VOLUME PRESERVING HYPERBOLIC DIFFEOMORPHISMS

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ABSTRACT. We introduce results concerning the number of ergodic components of a nonuniformly hyperbolic volume preserving C^∞ diffeomorphism or flow on a given connected compact Riemannian manifold. Such diffeomorphisms can have at most countably many ergodic components. We explain how to construct the diffeomorphisms and flows that is ergodic (i.e. with only one ergodic component) and that has infinitely many ergodic components.

0. INTRODUCTION

In this paper we discuss the following problem: For any given compact Riemannian manifold M, how many ergodic components can a smooth volume preserving hyperbolic diffeomorphism f has? Here *smooth* means C^∞, and *hyperbolic* means nonzero Lyapunov exponents almost everywhere with respect to the Lebesgue measure. Since ergodicity is a measure theoretic notion, we ignore sets of measure zero in the problem. So the number of ergodic components means the minimal number of ergodic components whose union is a full measure set. Also, we always assume that the manifold is connected and μ is the Lebesgue measure on M with $\mu M = 1$.

This is a natural question for nonuniformly hyperbolic systems. Let us recall the corresponding results for uniformly hyperbolic systems. It is well known that if a volume preserving diffeomorphism is uniformly hyperbolic on a compact invariant set, then it has only finitely many ergodic components up to a set of measure zero by the spectral decomposition theorem. In particular, if a diffeomorphism is uniformly hyperbolic on the whole manifold, then it is an Anosov system. However, it is known that not every manifold carries an Anosov diffeomorphism (see e.g. [F], [Mng]). It is also conjectured that the nonwandering set of an Anosov system is the whole manifold. The conjecture is true if the manifold is a torus or nilmanifold, or the system is codimension one (i.e. either $\dim E^u = 1$ or $\dim E^s = 1$) (see e.g. [F], [Mng] and [N]). In this case, the system has only one ergodic component.

A general result toward this direction for nonuniformly hyperbolic volume preserving diffeomorphism is due to Ya. Pesin. He proved in 1979 that any compact invariant set of such map can be decomposed into at most countably many ergodic components, plus a set of measure zero ([P]). Therefore our question can be split into two parts:

QUESTION. *Does every compact Riemannian manifold carry a smooth volume preserving hyperbolic diffeomorphism that (a) is ergodic, or (b) has infinitely many ergodic components?*

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Let us denote by m the number of ergodic components. It is easy to believe that if one can construct such examples for $m = 1$ and $m = \infty$, then similar methods can be applied to construct examples with any $1 < m < \infty$.

The first part of the question is answered affirmatively by A. Katok for $\dim M = 2$ in 1979 ([K]). In fact, he proved that the diffeomorphisms are Bernoulli. Using this result, M. Brin, J. Feldman and A. Katok showed that every manifold of dimension greater than two carries a Bernoulli diffeomorphism ([BFK]). However, the maps are not hyperbolic when the dimension of the manifold is greater than two, since the diffeomorphisms have only two nonzero Lyapunov exponents almost everywhere. Brin also showed that the diffeomorphisms can be constructed in such a way that they have all but one nonzero Lyapunov exponents if $\dim M \geq 5$ ([Br]).

It is easy to image that in Brin's construction, the zero Lyapunov exponent can be removed by a small perturbation. However, this is difficult to prove, because both the orbits and the directions of vectors that carry the Lyapunov exponent change. The study of stable ergodicity of partially hyperbolic systems, as well as the study of ergodic properties of nonuniformly hyperbolic systems, both of which have become active since 1990's, bring some new notions and ideas that help greatly to solve the problems we discuss.

In 2002, D. Dolgopyat and Ya. Pesin ([DP]) showed that the zero Lyapunov exponent can be removed by a perturbation. They also constructed examples for manifolds M with $\dim M = 3$ and 4. Therefore, the answer for part (a) of the above question is complete for all compact Riemannian manifold with $\dim M \geq 2$.

This result is also true for flows. That is, every compact Riemannian manifold of dimension greater than or equal to three carries a volume preserving hyperbolic flow whose time t map is ergodic for any $t \neq 0$ ([HPT]). Here hyperbolic flow means that all but one Lyapunov exponents are nonzero, since the tangent vectors to the flow must have zero Lyapunov exponent. We need $\dim M \geq 3$ because there is no hyperbolic area preserving flow on surfaces.

On the other hand, it was unknown till 2000 whether there exists a hyperbolic volume preserving diffeomorphisms with infinitely many ergodic components. The first example was constructed by D. Dolgopyat, H. Hu and Ya. Pesin on the 3-dimensional torus ([DHP]). Note that up to a measure zero set, points on the same stable and unstable manifolds belong to the the same ergodic components. In uniformly hyperbolic systems, the "size" of local stable and unstable manifolds is uniformly bounded from below. Hence the ergodic components can not be too small. On the other hand, in nonuniformly hyperbolic systems, the "size" of local stable and unstable manifolds can be arbitrarily small. Hence one can construct ergodic components with arbitrarily small volume.

Further, it was found out that hyperbolic volume preserving diffeomorphisms with infinitely many ergodic components can be arbitrarily close to the identity map in C^1 topology ([HT]). This is also different from uniformly hyperbolic systems, since it is generally believed that there is no uniformly hyperbolic (Anosov) system in a neighborhood of the identity.

It seems that by using the ideas for these results, one can construct a smooth volume preserving hyperbolic diffeomorphism with infinitely many ergodic components on any given compact Riemannian manifold of dimension greater than or equal to three. That is, the answer for part (ii) of above question should also be positive for manifolds of dimension three or higher.

Part 1. **General Results**

1. Upper bounds

In this section we introduce Pesin's result that a volume preserving hyperbolic diffeomorphism has at most countably many ergodic components. (See [P] or [BP].)

Theorem 1.1. *Let f be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold preserving the Lebesgue measure μ, and $\Lambda \subset M$ be an invariant set of positive measure. Then there exists invariant sets $\Lambda_0, \Lambda_1, \Lambda_2, \cdots$ such that*

1. $\bigcup_{i\geq 0} \Lambda_i = \Lambda$, *and* $\Lambda_i \cap \Lambda_j = \emptyset$ *whenever* $i \neq j$*;*
2. $\nu(\Lambda_0) = 0$, *and* $\nu(\Lambda_i) > 0$ *for each* $i \geq 1$*;*
3. *for each* $i \geq 1$, $f|_{\Lambda_i}$ *is ergodic, and* $f^{n_i}|_{\Lambda_i}$ *is Bernoulli for some* $n_i > 0$.

Proof. It is enough to show that any invariant set of positive Lebesgue measure contains an ergodic component of positive Lebesgue measure.

The proof consists two steps: use the Pesin's theory to select an invariant set of positive measure, and then use the Hopf's argument to prove that f is ergodic on the set. In the Hopf's argument, we also need the fact that the stable foliation $\mathcal{F}^s$ is absolutely continuous. That is, for any two nearby smooth submanifolds V_1 and V_2 of dimensional $\dim M - \dim \mathcal{F}^s$ transversal to $\mathcal{F}^s$, the holonomy or Poincaré map, defined by sliding from V_1 to V_2 along the stable leaves, sends measure zero sets in V_1 to measure zero sets in V_2. Note that the foliation $\mathcal{F}^s$ is only Hölder in general (see [HPS]), though the leaves in $\mathcal{F}^s$ are as smooth as f is. It takes some work to prove absolute continuity.

By Pesin's theory, up to a set of measure zero, one can write $\Lambda = \cup_{\ell>0}\Lambda^\ell$, where $\Lambda^\ell \subset \Lambda^{\ell+1}$ for any $\ell > 0$, such that on each Λ^ℓ, $Df_x|_{E^u_x}$ is uniformly expanding, and $Df_x|_{E^s_x}$ is uniformly contracting. Further, the size of local stable manifolds $V^s(x)$ and local unstable manifolds $V^u(x)$ at x are bounded from below on Λ^ℓ.

Let

$$P^\ell(x,r) = \bigcup_{y\in\Lambda^\ell\cap B(x,r)} (V^s(y) \cup V^u(y)).$$

Since $\Lambda = \cup_{\ell>0}\Lambda^\ell$, there exists $\ell > 0$, $x \in \Lambda^\ell$ such that $\mu P^\ell(x,r) > 0$ for some small $r > 0$. Let

$$Q(x) = \bigcup_{n\in\mathbb{Z}} f^n(P^\ell(x,r)).$$

It is an invariant set with positive Lebesgue measure.

Now we show that f is ergodic on $Q(x)$. We only need prove that for any L^1 function ϕ on $Q(x)$, the time average $\phi^+(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k x)$ is constant almost everywhere. Since continuous functions are dense in the space of L^1 functions, we may assume that ϕ is a continuous function. Further, by invariance, we only need to show that $\phi^+(x)$ is constant for almost every $x \in P^\ell(x,r)$.

Denote $\phi^-(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^{-k}x)$. Let $B = \{y \in P^\ell(x,r) : \phi^+(y) = \phi^-(y)\}$. By Birkhoff's ergodic theorem, $\mu B = \mu P^\ell(x,r)$. So we can find a unstable leaf $V^u(y)$ such that with respect to μ_y, the Lebesgue measure restricted to $V^u(y)$, almost every point z in $V^u(y)$ belongs to B, i.e. $\mu_y V^u(y) = \mu_y(V^u(y) \cap B)$.

Since ϕ is continuous and for any $z \in V^u(y)$, $d(f^{-n}z, f^{-n}y) \to 0$, we have $|\phi(f^{-n}z) - \phi(f^{-n}y)| \to 0$ as $n \to \infty$. Hence $\phi^-(z) = \phi^-(y)$. Similarly, for any $z' \in V^s(z)$, $\phi^+(z') = \phi^+(z)$. So if $z \in B$, then for any $z' \in V^s(z)$,

$$\phi^+(z') = \phi^+(z) = \phi^-(z) = \phi^-(y).$$

Now we get that (i) for μ_y-a.e $z \in V^y$, $\phi^+(z) = \phi^-(z) = \phi^-(y)$ and (ii) for any $z' \in V^s(z)$, $\phi^+(z') = \phi^-(y)$. Combining (i) and (ii) with absolute continuity of the stable foliation, we get that for almost every $z' \in P^\ell(x,r)$, $\phi^+(z') = \phi^-(y)$.

The proof of the part that $f^{n_i}|_{\Lambda_i}$ is Bernoulli is based on the same idea in [OW]. The main part is to construct a "very week Bernoulli" partition into rectangles. Then use a result of Ornstein and Weiss which says that a system is Bernoulli if it has a sequence of increasing very week Bernoulli partitions $\xi_n \nearrow \epsilon_0$, a partition into single points. We refer [P] and [OW] for details. $\square$

2. Ergodicity and Bernoullicity

In this section we introduce some general conditions under which a smooth volume preserving diffeomorphism is ergodic or Bernoulli.

The following result can be seen in [P], Theorem 7.8 or [BP], Theorem 14.5. We say that a set Λ is open $(\mathrm{mod}\, 0)$, if there is a set Z of measure zero such that $\Lambda \triangle Z$ is an open set.

Theorem 2.1. *Let f be a volume preserving diffeomorphism of a compact manifold M and Λ an f invariant hyperbolic subset with $\mu\Lambda > 0$. Suppose every ergodic component of positive measure lying in Λ is open* $(\mathrm{mod}\, 0)$*. Then f is ergodic whenever it is topologically transitive.*

Proof. Assume that f has two ergodic components C and D of positive measure. Then for all $n > 0$, $\mu(f^nC \cap D) = 0$. By topological transitivity, there is $n > 0$ such that $f^nC \cap D \neq \emptyset$. Since both C and D are open $(\mathrm{mod}\, 0)$, so is $f^nC \cap D$. We have $\mu(f^nC \cap D) > 0$, a contradiction. $\square$

A sufficient condition for an ergodic component to be open $(\mathrm{mod}\, 0)$ is that the stable or unstable foliation of the system forms a C^1 *continuous lamination*. Roughly speaking, it means that there are continuous positive function δ and q on Λ, such that for every $x \in \Lambda$, the family of embeddings $\Phi_y : D \to M$ that send a disk of dimension $\dim E^s_y$ or $\dim E^u_y$ to the local stable or unstable manifold of size $\delta(x)$ respectively is C^1 continuous with y for all $y \in \Lambda \cap B(x, q(x))$.

Proposition 2.2. *Let Λ be a compact hyperbolic invariant subset of a volume preserving diffeomorphism f. If the stable or unstable foliation forms a C^1 continuous lamination of Λ, then every ergodic component of positive measure is open* $(\mathrm{mod}\, 0)$ *in Λ with respect to the induced topology.*

Proof. Suppose that the stable foliation forms a C^1 continuous lamination. Then for any $x \in \Lambda$, the set $\displaystyle\bigcup_{y \in W^u(x,q(x))} W^s(y, \delta(y))$ contains a open neighborhood of x. By the Hopf's argument, almost every point in the set belongs to the same ergodic component. $\square$

Now we consider partially hyperbolic case. A diffeomorphism f is *partially hyperbolic* if there is an f-invariant decomposition of the tangent bundle into

$$TM = E^u \oplus E^c \oplus E^s$$

and constants $a < b < 1 < c < d$ such that for all $x \in M$,

$$\begin{aligned} \|Df_x(v)\| &\le a\|v\| && \forall v \in E^s, \\ b\|v\| \le \|Df_x(v)\| &\le c\|v\| && \forall v \in E^c, \\ d\|v\| &\le \|Df_x(v)\| && \forall v \in E^u. \end{aligned}$$

The bundles E^s and E^u are called the stable and unstable bundles respectively, and E^c the center bundle.

There is a conjecture given by C. Pugh and M. Shub, called *stable ergodicity conjecture*, related to our problem with $m = 1$. (See e.g. [PS], Conjecture 1, also [BPSW] Conjecture 0.1.)

Conjecture 2.3. *On any compact manifold, ergodicity holds for an open and dense set of C^2 volume preserving partially hyperbolic diffeomorphisms.*

We should mention that the answer of the conjecture does not give an answer of the ergodicity part in our question, because not every compact manifold admits a partial hyperbolic system, such as the three dimensional sphere.

We say that two points $x, y \in M$ are *accessible* if they can be joined by a piecewise differentiable piecewise nonsingular path which consists of segments tangent to either E^u or E^s. The diffeomorphism f is *essentially accessible* if almost any two points in M (with respect to the Riemannian volume) are accessible. By the Hopf's argument, almost every point in the same stable or unstable leaf lies in the same ergodic component. So it is natural to have the following ([PS] and [BPSW]).

Conjecture 2.4. *A partially hyperbolic C^2 volume preserving diffeomorphism with the essential accessibility property is ergodic.*

With some additional conditions the conjecture is proved by C. Pugh and M. Shub ([PS], Theorem A).

Theorem 2.5. *Let f be a C^2 diffeomorphism of a compact Riemannian manifold preserving the Lebesgue measure μ. If f is center bunched, partially hyperbolic, dynamically coherent, and essentially accessible, then f is ergodic.*

Here we say that f is *center bunched* if both $\|Df|_{E^c}\|$ and $\|Df^{-1}|_{E^c}\|$ are sufficiently close to 1. We say that f is *dynamically coherent* if the distributions E^c, $E^{cs} := E^c \oplus E^s$ and $E^{cu} := E^c \oplus E^u$ are integrable, and the central foliation $\mathcal{F}^c$ subfoliates both the central-stable foliation $\mathcal{F}^{cs}$ and the central-unstable foliation $\mathcal{F}^{cu}$.

The first part of the following results is from [PS], Corollary of Theorem 2.3 (see also [BPSW] Proposition 3.1). The second part is a consequence of Theorem 2.5 and the facts that partial hyperbolicity and center bunching are C^1 open properties.

Theorem 2.6. *Suppose that f is partially hyperbolic, and has a C^1 central foliation $\mathcal{F}^c$, then any f' close enough to f in C^1 topology is dynamically coherent. Therefore, if f' is essentially accessible, then it is ergodic.*

The diffeomorphisms we will introduce, when restricted to the ergodic components, are all Bernoulli. A system is said to be *Bernoulli* if it is measure theoretically isomorphic to a Bernoulli process. Bernoulli systems enjoy all the other ergodic properties, including ergodicity, mixing and multiple mixing, Kolmogorov property.

For hyperbolic systems, Bernoullicity can be obtained by using Theorem 1.1.(3).

Corollary 2.7. *Under the assumptions of the Theorem 1.1, if $f^n : \Lambda \to \Lambda$ is ergodic for any $n > 0$, then $f|_\Lambda$ is a Bernoulli diffeomorphism.*

If a system is not hyperbolic, them we need additional conditions. The following result is proved by N. Chernov ([C]).

Theorem 2.8. *Let f^t be a suspension flow of a hyperbolic system with a C^2 ceiling function. If for any $t \neq 0$, the time t map has the Kolmogorov property, then the map is Bernoulli.*

It is known that a system has the Kolmogorov property if and only if the Pinsker algebra is trivial. We refer [W] for the definitions and reasons.

3. Reduce the problem to some particular manifolds

In this section we explain that to construct volume preserving hyperbolic diffeomorphisms with given number of ergodic components on *any* given manifold M, we only need do so on *some* manifold N, if N has the same dimension with M and can be embedded into $\mathbb{R}^n$, and if the diffeomorphisms can be made to be "sufficiently flat" near the boundary of N.

A measure is *smooth* if it is absolutely continuous with respect to the Lebesgue measure, and the density is a smooth function. Recal that μ is the Lebesgue measure on M with $\mu M = 1$. The proof of the next proposition can be seen in [K].

Proposition 3.1. *For any compact n-dimensional Riemannian manifold M (possibly with boundary) and a smooth probability measure ν on $\mathcal{D}^2$, there is a continuous mapping $\tau : \mathcal{D}^n \to M$ with the following properties:*

1) $\tau|_{\text{int}\,\mathcal{D}^n}$ *is a diffeomorphic embedding;*
2) $\tau(\mathcal{D}^n) = M$*;*
3) $\mu(M \backslash \tau(\text{int}\,\mathcal{D}^n)) = 0$*;*
4) $\tau_* \mu = \nu$.

Proof. Let $\sigma_1, \cdots, \sigma_l$ be the n-dimensional simplexes of triangulation, so ordered that each σ_k has a common $(n-1)$-dimensional face with some σ_j, $j < k$.

Denote $M_k = \cup_{i=1}^k \sigma_i$. Observe that if $\tau_{k-1} : \mathcal{D}^n \to M_{k-1}$ is a map satisfying the requirement 1)-3) with M replaced by M_{k-1}, and σ_j, $j < k$ has a common face with σ_k, then we can construct $\tau_k : \mathcal{D}^n \to M_k$ by making $\tau_k = \tau_{k-1}$ if $\tau_{k-1}(x) \notin \sigma_j$, and otherwise $\tau_k = \tau_k' \circ \tau_{k-1}$ for some suitable diffeomorphism $\tau_k' : \sigma_j \to \sigma_k \cup \sigma_j$ such that the requirement 1)-3) are still satisfied for τ_k. By induction we get a map $\widehat{\tau} := \tau_l : \mathcal{D}^n \to M$ satisfying all the requirements but 4).

To obtain 4), we can take a diffeomorphism $\delta : \mathcal{D}^n \to \mathcal{D}^n$ such that $\delta_*(\widehat{\tau}_* \mu) = \nu$, where $\widehat{\tau}_*$ is given by $\widehat{\tau}_* \mu(A) = \mu(\widehat{\tau}^{-1} A)$. Such δ exists because of the Moser theorem (see [Ms], also see [KH], Theorem 5.1.27), which says that for any two volume forms Ω_1 and Ω_2 on a manifold with the same total volume, there is a diffeomorphism δ such that $\delta^* \Omega_1 = \Omega_2$. Now we put $\tau = \widehat{\tau} \circ \delta$, then $\tau_* \mu = \delta_*(\widehat{\tau}_* \mu) = \nu$. □

Proposition 3.2. *The results in the above proposition is true if we replace $\mathcal{D}^n$ by any compact Riemannian manifold N of dimension n that can be embedded in $\mathbb{R}^n$.*

Proof. Since N can be embedded in $\mathbb{R}^n$, we may assume that $N \subset \mathbb{R}^n$. Observe that we can identify some points on the boundary of N such that the resulting manifold is diffeomorphic to the ball $\mathcal{D}^n$. So we get a map $\widetilde{\tau}$ that satisfies the requirement 1)-3) in Proposition 3.1 with $\mathcal{D}^n$ and M replaced by N and $\mathcal{D}^n$ respectively. Now the map $\widetilde{\tau}$ composed with the map in Proposition 3.1, and composed with another map from the Moser theorem if necessary, is the desired map. $\square$

If we apply the above proposition to a diffeomorphism $f : N \to N$ to get a smooth dynamical system $\widehat{f} : M \to M$ for a given manifold M, we need that f is "sufficiently flat" near the boundary ∂N. It can be described in the following way (See [K]). We may assume $N \subset \mathbb{R}^n$, and therefore a diffeomorphism f of N has the form $f = (f_1, \cdots, f_n)$. Let $\rho = (\rho_0, \rho_1, \cdots)$ be a sequence of real-valued continuous functions on N. It is called *admissible* if every function ρ_i is nonnegative and is strictly positive inside N. We denote by $C^\infty_\rho(N)$ the set of functions $\alpha \in C^\infty(N)$ such that the absolute values of all partial derivatives of α of order k are bounded by ρ_k on N. Then we denote by $\mathrm{Diff}^\infty_\rho(N)$ the set of diffeomorphisms $f = (f_1, \cdots, f_n) \in \mathrm{Diff}^\infty(N)$ such that $f_i - x_i \in C^\infty_\rho(N)$ for all i.

Suppose that for some N we can construct a volume preserving hyperbolic diffeomorphism $f \in \mathrm{Diff}^\infty_\rho(N)$ for some admissible sequence ρ. For any given manifold M with $\dim M = \dim N$, by the above proposition we can take a map $\tau : N \to M$, and then define

$$\widehat{f}(x) = \begin{cases} \tau(f(\tau^{-1}x)) & \text{if } x \in \tau(\mathrm{int}\, N); \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\bar{f} : M \to M$ is well defined. Also, $\bar{f}$ is volume preserving and hyperbolic, and has the same number of ergodic components as f does up to a set of measure zero. Since $f \in \mathrm{Diff}^\infty_\rho(N)$, $\widehat{f}$ is a smooth diffeomorphism on M.

Part 2. Constructions

4. Ergodic diffeomorphism on surfaces

In this section we introduce Katok's construction for two dimensional case. This is also the important part for constructing examples in higher dimensional spaces.

Theorem 4.1. *For any admissible sequence of functions ρ on $\mathcal{D}^2$, there exists an area preserving hyperbolic Bernoulli diffeomorphism $g \in \mathrm{Diff}^\infty_\rho(\mathcal{D}^n)$.*

Proof. We start with the linear automorphism g_0 induced by the matrix $\begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$. The map has four fixed points

$$x_1 = (0,0)\,,\ x_2 = \left(\frac{1}{2}, 0\right)\,,\ x_3 = \left(0, \frac{1}{2}\right)\,,\ x_4 = \left(\frac{1}{2}, \frac{1}{2}\right).$$

The construction consists four steps.

1) Slow down the motion around the fixed points to get a diffeomorphism g_1 such that $Dg_1(x_i) = \mathrm{id}$. g_1 can be made conjugate to g_0, and preserves a smooth invariant measure ν.
2) Change the metric near the fixed points to make the map area preserving. This can be down by using a map ϕ_1 on $\mathcal{T}^2$.
3) Take a double branched covering $\phi_2 : \mathcal{T}^2 \to \mathcal{S}^2$. The resulting system is on $\mathcal{S}^2$, and is hyperbolic everywhere except at x_i.
4) Blow up the point x_4 into a circle by a map ϕ_3 to get a system on $\mathcal{D}^2$.

The construction can be represented in the following commutative diagram.

$$\begin{array}{ccccccccc}
(\mathcal{T}^2,\mu) & \xrightarrow{h} & (\mathcal{T}^2,\nu) & \xrightarrow{\phi_1} & (\mathcal{T}^2,\mu) & \xrightarrow{\phi_2} & (\mathcal{S}^2,\mu) & \xrightarrow{\phi_3} & (\mathcal{D}^2,\mu) \\
\downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & {\scriptstyle g}\downarrow \\
(\mathcal{T}^2,\mu) & \xrightarrow{h} & (\mathcal{T}^2,\nu) & \xrightarrow{\phi_1} & (\mathcal{T}^2,\mu) & \xrightarrow{\phi_2} & (\mathcal{S}^2,\mu) & \xrightarrow{\phi_3} & (\mathcal{D}^2,\mu)
\end{array}$$

Construction of g_1. Choose a C^∞ function $\psi : [0,1] \to \mathbb{R}$ such that $\psi(0) = 0$, $\psi(u) = 1$ for $u > r$ where $r \in (0,1)$ is small, $\psi'(u) \geq 0$, and

$$\int_0^1 \frac{du}{\psi(u)} < \infty. \tag{4.1}$$

Note that near x_i the map g_0 is the time one map of a flow generated by the vector field $(\dot{s}_1, \dot{s}_2) = ((\ln\alpha)s_1, -(\ln\alpha)s_2)$, where α is the larger eigenvalue of the matrix generating g_0. We let g_1 be the the time one map of the flow generated by $(\dot{s}_1, \dot{s}_2) = ((\ln\alpha)s_1\psi(s_1^2+s_2^2), -(\ln\alpha)s_2(s_1^2+s_2^2))$ in a small neighborhood of each x_i, and let $g_1 = g_0$ otherwise.

Clearly, $Dg_1(x_i) = \mathrm{id}$. It can be proved that g_1 is topologically conjugate to g_0 with some conjugacy h that transfers the stable and unstable manifolds of g_0 into smooth curves. Hence, g_1 is nonuniformly hyperbolic. Also, it is easy to see that g_1 preserves a smooth measure with density $\rho(x) = 1/\psi(s_1^2(x) + s_1^2(x))$ for x close to x_i and $\rho(x) = 1$ otherwise. The condition in (4.1) guarantees that the density function is integrable. Denote $\rho_0 = \int_{\mathcal{T}^2} \rho d\mu$.

To get smoothness of g_2 and g_3, and to make $g \in \mathrm{Diff}^\infty_\rho(\mathcal{D}^2)$, we first note that all g_i and g are locally time one map of vector fields near x_i and $\partial\mathcal{D}^2$. Let H_i and H be the corresponding Hamiltonian. Then $H_0 = H_1$, $H_2 = H_1 \circ \phi_1^{-1}$, $H_3 = 2H_2 \circ \phi_2^{-1}$ and $H = H_3 \circ \phi_3^{-1}$. Clearly, $H_0(s_1, s_2) = (\ln\alpha)s_1 s_2$ near x_i. If we let $\beta(u)$ be the inverse to $\gamma(u) = \Big(\int_0^u \frac{d\tau}{\psi(\tau)}\Big)^{1/2}$, then we have

$$H_2(s_1,s_2) = \frac{(\ln\alpha)s_1 s_2 \beta(\sqrt{s_1^2+s_2^2})}{s_1^2+s_2^2}, \qquad H_3(\tau_1,\tau_2) = \frac{(\ln\alpha)\tau_2\beta(\sqrt{\tau_1^2+\tau_2^2})}{\sqrt{\tau_1^2+\tau_2^2}},$$

$$H(x_1,x_2) = \frac{(\ln\alpha)x_2\beta(\sqrt{1-x_1^2-x_2^2})}{\sqrt{x_1^2+x_2^2}}.$$

for H_2 and H_3 near x_i and for H near $\partial\mathcal{D}^2$.

If the derivatives of $\beta(u)$ decrease near zero sufficiently fast, then H_2, H_3 and H can decrease at any given speed.

Construction of ϕ_1. ϕ_1 is given by

$$\phi_1(s_1,s_2) = \frac{1}{\sqrt{\rho_0(s_1^2+s_2^2)}} \left(\int_0^{s_1^2+s_2^2} \frac{du}{\psi(u)}\right)^{1/2} (s_1,s_2)$$

near x_i and it is the identity otherwise.

It is easy to check that $\det D\phi_1(s_1,s_2) = 1/\rho_0(s_1^2+s_2^2)$. So g_2 preserves the Lebesgue measure μ on $\mathcal{T}^2$. Note that derivatives of $\beta(u)$ decrease near zero sufficiently fast. So it is also easy to see $Dg_2(x_i) = \mathrm{id}$ by the expression of H_2.

Construction of ϕ_2. ϕ_2 is a double branched covering, and is regular and C^∞ everywhere except for the points x_i. It commutes with the involution $J(t_1,t_2) =$

$(1-t_1, 1-t_2)$, and preserves the Lebesgue measure. Near x_i it has the form

$$\phi_2(s_1, s_2) = \frac{1}{\sqrt{s_1^2 + s_2^2}} \left(s_1^2 - s_2^2, s_1 s_2\right).$$

Construction of ϕ_3. Near x_4, ϕ_3 has the form

$$\phi_3(\tau_1, \tau_2) = \frac{\sqrt{1 - \tau_1^2 - \tau_2^2}}{\sqrt{\tau_1^2 + \tau_2^2}} (\tau_1, \tau_2),$$

and then it is is extended to a diffeomorphism from $\mathcal{S}^2 \backslash \{x_4\}$ to $\operatorname{int} \mathcal{D}^2$ which preserves the Lebesgue measure.

It can be proved that g has continuous lamination, and is topologically transitive. So by Proposition 2.2 and Theorem 2.1 g is ergodic. Since this is also true for any g^n, $n > 0$, then by Corollary 2.7, f is Bernoulli. □

Corollary 4.2. *There is a neighborhood $U \subset \mathcal{D}^2$ of $\partial\mathcal{D}^2$ such that the map $g|_U : U \to \mathcal{D}^2$ can be embedded into an area preserving flow.*

Proof. This is because in the above construction, g_1 is generated by a flow and ϕ_i, $i = 1, 2, 3$, send a flow to a flow. The area preserving part for the flow follows from the same arguments for the map g. □

By Theorem 4.1 and the discussion in the previous section, we get the following ([K], Theorem B).

Theorem 4.3. *Every smooth compact surface carries a C^∞ area preserving hyperbolic Bernoulli diffeomorphism.*

5. Ergodic diffeomorphism on any given manifold

In this section we introduce Brin's construction [Br] of volume preserving ergodic diffeomorphisms on any given manifold M with $\dim M \geq 5$ that has all but one nonzero Lyapunov exponents.

Let $n \geq 5$ and $k = [(n-3)/2]$. Consider a linear automorphism from $\mathcal{T}^{n-3}$ to itself induced by a matrix $A = \operatorname{diag}\{A_1, \cdots, A_k\}$, where $A_i = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ if $i < k$ or $i = k$ and n is odd, and $A_k = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ if n is even.

We also denote by A the linear automorphism of $\mathcal{T}^{n-3}$ induced by the matrix. The suspension manifold of the linear automorphism A is the manifold $L = \mathcal{T}^{n-3} \times [0,1]/\sim$, where $\sim$ is the identification $(x, 1) = (Ax, 0)$.

Proposition 5.1. *The suspension manifold of the linear automorphism A from $\mathcal{T}^{n-3}$ to itself can be embedded into the space $\mathbb{R}^{n-1} \times \mathcal{S}^1$.*

Proof. Note that $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. We first claim that there is a deformation $\mathcal{T}_t^2$ of the two dimensional torus in $\mathbb{R}^3 \times \mathcal{S}^1$ which transforms $\mathcal{T}_0^2$ to its image under $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Take strips $D_0 = \mathcal{S}^1 \times \{(x_1, 0, 0) : 0 < x_1 < 1\}$ and $D_1 = \{(\phi, r\cos\phi, r\sin\phi, 0) : \phi \in \mathcal{S}^1, 0 < r < 1\}$, both in $\mathcal{S}^1 \times \mathcal{D}^3$. We construct a deformation from D_0 to D_1. Let $I(\phi, t)$ be the line segment in $\mathcal{S}^1 \times \mathcal{D}^3$ from the point $(\phi, 0, 0, 0)$ to

$$(\phi, x_1, x_2, x_3) = \Big(\phi,\ \cos^2\frac{\pi t}{2} + \sin^2\frac{\pi t}{2}\cos\phi,\ \sin\frac{\pi t}{2}\sin\phi,\ \frac{1}{2}\sin\pi t(1-\cos\phi)\Big).$$

Note that for fixed $\phi \in \mathcal{S}^1$, the terminate point of $I(\phi, t)$ change from $(1, 0, 0)$ to $(\cos\phi, \sin\phi, 0)$ on $\mathcal{S}^2 = \partial\mathcal{D}^3$ as t varies from 0 to 1. So the strips $D_t = \{I(\phi, t) : \phi \in \mathcal{S}^1\}$ give the deformation. We also note that for fixed $t \in [0, 1]$, the terminate points of $I(\phi, t)$ for $\phi \in \mathcal{S}^1$ form a circle on the surface $\mathcal{S}^2$ of radius $\sin\dfrac{\pi t}{2}$ and centered at the point $\left(\cos^2\dfrac{\pi t}{2},\ 0,\ \cos\dfrac{\pi t}{2}\sin\dfrac{\pi t}{2}\right)$.

Since D_t has trivial normal bundle, there exist two vector fields $v_1(t, \phi)$ and $v_2(t, \phi)$ normal to D_t. Let $\hat{v}_1(t, \phi)$ be the restriction of $v_1(t, \phi)$ to the middle point of $I(\phi, t) \subset D_t$. We take the circle $C(\phi, t)$ in $\{\phi\} \times \mathcal{S}^3$ for each ϕ and t that is determined by $\hat{v}_1(t, \phi)$ and $I(\phi, t)$. So for each t we get a torus $\mathcal{T}_t = \{C(\phi, t) : \phi \in \mathcal{S}^1\}$. Further, we can choose $v_1(t, \phi)$ in such a way that $\mathcal{T}_1$ is the image of $\mathcal{T}_0$ under the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. So the claim is true.

The matrix A can be written as a product of j matrices of the forms $A^{(l)} = \mathrm{diag}\{E_i, \hat{A}, E_{n-5-i}\}$, where $j = [n/2] - 1$, E_i is the unit matrix of order i, and $\hat{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. The above claim implies that for each $A^{(l)}$ there is an embedding from $\mathcal{T}^{n-3} \times [(l-1)/j, l/j]$ to $\mathbb{R}^{n-1} \times \mathcal{S}^1$ such that $\mathcal{T}^{n-3} \times \{l/j\}$ is the image of $\mathcal{T}^{n-3} \times \{(l-1)/j\}$ under $A^{(l)}$. Repeating the arguments j times, and identifying $\mathcal{T}^{n-3} \times \{0\}$ with $\mathcal{T}^{n-3} \times \{1\}$, we get the result of the proposition. □

Theorem 5.2. *For every smooth compact manifold M of dimensional $n \geq 5$ there is a volume preserving Bernoulli diffeomorphism which has $n-1$ nonzero Lyapunov exponents.*

Proof. Take an $(n-3) \times (n-3)$ matrix A as in the beginning of the the section. By Proposition 5.1, we may assume that the suspension flow of the linear automorphism A is given by $h^t : L \to L$, where L is an $n-2$ dimensional manifold in $\mathbb{R}^n$. Note that L is orientable codimension two manifold and has trivial normal bundle. Hence $N := \mathcal{D}^2 \times L$ is a manifold still in $\mathbb{R}^n$ with boundary $\partial N = \partial\mathcal{D}^2 \times L$.

Let $g : \mathcal{D}^2 \to \mathcal{D}^2$ be the diffeomorphism given in Theorem 4.1. Take a function $\alpha \in C^\infty_\rho(\mathcal{D}^2)$ for some admissible sequence ρ. (Recall that $C^\infty_\rho(N)$ is introduced in Section 3.) Then we define a map $f : N \to N$ by

$$f(x, y) = (g(x), h^{\alpha(x)}(y)).$$

It can be proved that f has trivial Pinsker algebra if $\alpha \not\equiv 0$ (see [Br]). So f has the Kolmogorov property and therefore is ergodic. To obtain that f is Bernoulli, we can either use similar argument as in [P] (see [Br]), or use the fact that f is a time one map of some suspension flow of a hyperbolic system and then use Theorem 2.8.

We can see that f is volume preserving. f has only one zero Lyapunov exponent, which is along the flow direction of h^t.

The result is true for any compact manifold M by the discussion in Section 3. □

6. Ergodic hyperbolic diffeomorphisms on any given manifold

Pesin and Dolgopyat obtained ergodic volume preserving hyperbolic diffeomorphisms by removing the zero Lyapunov exponent in Brin's construction. They also obtained examples for $\dim M = 3$ and 4 (see [DP]).

Theorem 6.1. *For every smooth compact manifold M of dimensional $n \geq 3$ there is a volume preserving hyperbolic Bernoulli diffeomorphism.*

Proof. For $n \geq 5$, one can start with the diffeomorphism $f : N \to N$ constructed in Theorem 5.2, and then make a perturbation to remove the zero Lyapunov exponent. The perturbation is a rotation along the center and stable directions in a small ball.

Recall that in Katok's diffeomorphism $g : \mathcal{D}^2 \to \mathcal{D}^2$, outside a neighborhood U of x_1, x_2, x_3 and $\partial\mathcal{D}^2$, g is uniformly hyperbolic with hyperbolicity λ. For any small γ we choose a point $x_0 \in \mathcal{D}^2$ such that $g^j B(x_0, \gamma)$ disjoint with $B(x_0, \gamma)$ and U for all j with $|j| \leq n_0 := -(\log\gamma/\log\lambda) - C$, where C is a constant independent of ϵ. Then take $y_0 \in L$ and let $z_0 = (x_0, y_0) \in N$. Near z_0 we take a coordinate system $z = (x, y) = (x_1, x_2, y_1, y_2, \cdots y_{n-2})$ such that $\dfrac{\partial}{\partial y_1} \in E_z^c$ and $\dfrac{\partial}{\partial y_2} \in E_z^s$. Let

$$\begin{aligned}\phi(x,y) = (x_1, x_2, &y_1 \cos\epsilon\psi(\tau) - y_2 \sin\epsilon\psi(\tau), \\ &y_1 \sin\epsilon\psi(\tau) + y_2 \cos\epsilon\psi(\tau), y_3, \cdots, y_{n-2}),\end{aligned} \tag{6.1}$$

where ψ is a C^∞ nonnegative function with compact support, and $\tau = \gamma^{-2}(\|x\|^2 + \|y\|^2)$. Let $\widetilde{f} = \phi \circ f$ be the perturbed diffeomorphism.

It is clear that $\widetilde{f}$ is volume preserving. To get hyperbolicity, we first show that

$$\int \log|\det D\widetilde{f}|_{E^c(\widetilde{f})}|d\mu < 0 \tag{6.2}$$

By the choice of ϕ, we know that

$$\det D\phi|_{E^s(z,f)} = -\frac{2y_1y_2}{\gamma^2}\epsilon\psi'(\tau)\cos(\epsilon\psi(\tau)) + \cos(\epsilon\psi(\tau)) - \frac{2y_2^2}{\gamma^2}\epsilon\psi'(\tau)\cos(\epsilon\psi(\tau)).$$

It can be computed (see page 424 in [DP] for details) that there is $C > 0$ such that

$$\int_{B(z_0,\gamma)} \log|\det D\phi|_{E^s(z,f)}| = -C\epsilon^2\gamma^{n-2} + O(\gamma^{n-2}\epsilon^3). \tag{6.3}$$

Note that $\widetilde{f}$ is close to f in C^1 topology. So $\forall \tilde{z} \in N$, $d(E^s(\tilde{z}, f), E^s(\tilde{z}, \widetilde{f})) \leq \delta$ for some $\delta = \delta(\epsilon) > 0$. In particular, for any $z \in B(z_0, \gamma)$, we can take $\tilde{z} = f^{n_0}z$. By hyperbolicity, there is $\zeta > 1$ such that $d(E^s(\tilde{z}, f), E^s(\tilde{z}, \widetilde{f})) \leq \delta\zeta^{-n_0} \leq C\delta\gamma^{\log\zeta/\log\lambda}$, where we use the fact $n_0 = -(\log\gamma/\log\lambda) - C$.

Now we let F and $\widetilde{F}$ be the first return maps of f and $\widetilde{f}$ on the set $B(z_0, \gamma)$ respectively. The above fact implies

$$|\det D\widetilde{F}(z)|_{E^s(z,\widetilde{f})}| \leq |\det D\phi|_{E^s(Fz,f)}| \cdot |\det DF(z)|_{E^s(z,f)}| \cdot (1 + O(\gamma^{\log\zeta/\log\lambda})).$$

Hence, by (6.3)

$$\begin{aligned}&\int_{B(z_0,\gamma)} \log|\det D\widetilde{F}(z)|_{E^s(z,\widetilde{f})}|d\mu - \int_{B(z_0,\gamma)} \log|\det DF(z)|_{E^s(z,f)}|d\mu \\ &= -C\epsilon^2\gamma^{n-2} + O(\gamma^{n-2}\epsilon^3) + O(\gamma^{n+\log\zeta/\log\lambda}).\end{aligned} \tag{6.4}$$

So if we take γ and ϵ such that $\gamma^2 \leq e^3$, then the right side is strictly negative.

Note that $\int \log|\det Df|_{E^u_f}|d\mu + \int \log|\det Df|_{E^c_f}|d\mu + \int \log|\det Df|_{E^s_f}|d\mu = 0$ and the same is also true for $\widetilde{f}$ since they are volume preserving. Since in the perturbation, the foliation $\mathcal{F}^{cs}$ is preserved, we have $\int \log\left|\det Df|_{E^u(f)}\right| d\mu = \int \log|\det D\widetilde{f}|_{E^u(\widetilde{f})}|d\mu$, though the subbundles $E^u(f)$ and $E^u(\widetilde{f})$ are different in general. Since $\int \log|\det Df|_{E^c(f)}|d\mu = 0$, by (6.4) we get (6.2).

Let Λ be the set of points that does not have zero Lyapunov exponents for $\widetilde{f}$. It is an $\widetilde{f}$-invariant set. (6.2) implies $\mu\Lambda > 0$. Since $\widetilde{f}$ is hyperbolic on Λ and has continuous lamination, by Proposition 2.2 the ergodic components of Λ is open $(\mathrm{mod}\, 0)$. Also, it can be proved that almost every point in N has dense orbit. It follows that $\mu\Lambda = 1$ and therefore $\widetilde{f}$ is ergodic. Further, the same arguments also work for all $\widetilde{f}^n$ if $n > 0$. By Corollary 2.7, $\widetilde{f}$ is Bernoulli.

For the case $\dim M = 3$ and 4, take $N = \mathcal{D}^2 \times \mathcal{S}^{\dim M - 2}$, and let $f(x, y) = (g(x), r_{\alpha(x)}(y))$, where g is the map given by Katok's construction, r_t is a rotation by t and and $\alpha \in C^\infty_\rho(\mathcal{D}^2)$ for some admissible sequence ρ. Clearly, f is volume preserving. Note that f has $\dim M - 2$ zero Lyapunov exponents. To remove the first one, we put a rotation similar as in (6.1) along the stable direction and one center direction.

For the case $\dim M = 4$, now the system has a small negative exponent in direction E^{c_1} and a zero exponent in direction E^{c_2}. It is unclear whether the same method can remove the second zero exponent in E^{c_2}. Instead, we use a sequence of rotations to rotate vectors in E^{c_2} into E^{c_1}, each of which rotate vectors by a small angle. Then it can be proved that vectors in both E^{c_2} and E^{c_1} have about the same exponents. This method is also used in [Mñ] and [Bch].

In both case, ergodicity and Bernoullicity can be obtained by the same arguments for the case $\dim M \geq 5$. $\square$

Since $D\widetilde{f}|_{\partial N} = \mathrm{id}$, the diffeomorphism constructed has uncountably many indifferent fixed points. Recall that on the sphere, Katok's diffeomorphism g_3 is hyperbolic everywhere except at finite number of fixed points. It seems that such diffeomorphisms can be constructed on other compact surfaces (See e.g. [GK]). So we have a natural question:

Question. *Does every compact Riemannian manifold carries a volume preserving ergodic diffeomorphism that is hyperbolic everywhere except at finitely many fixed points?*

The hyperbolicity part means that there exists a decomposition $TM = E^u \oplus E^s$ of the tangent bundle such that $Df|_{E^u}$ is strictly expanding and $Df|_{E^s}$ is strictly contracting except at finitely many fixed points.

7. Hyperbolic ergodic flow on any given manifold

The results in the above section also hold for flow. Such examples are obtained by H. Hu, Ya. Pesin and A. Taliskaya ([HPT]) for any manifold of dimension greater than or equal to three. This is because there is no area preserving hyperbolic flow on any compact surface.

We say that a flow is Bernoulli, if its time t map for nay $t \neq 0$ is Bernoulli.

Theorem 7.1. *For every smooth compact Riemannian manifold M of dimensional $n \geq 3$ there is a volume preserving hyperbolic Bernoulli flow.*

Proof. First we consider the case $\dim M \geq 5$. Take a map $g : \mathcal{D}^2 \to \mathcal{D}^2$ as the Katok's diffeomorphism, and an automorphism $A : \mathcal{T}^{n-3} \to \mathcal{T}^{n-3}$ as in the Brin's construction. Then take the suspension flow $\sigma_R : K \to K$ of the map $R = g \times A$, where $K = \mathcal{D}^2 \times \mathcal{T}^{n-3} \times [0,1]/\sim$, and $\sim$ is the identification $(x, y, 1) \sim (gx, Ay, 0)$. Let Z be the vector field of the flow. For each point $(x, y, t) \in K$, we take a local coordinate system $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \cdots \frac{\partial}{\partial y_{n-3}}, \frac{\partial}{\partial t}\right)$ in its neighborhood, where $x = (x_1, x_2) \in \mathcal{D}^2$, $y = (y_1, \dots y_{n-3}) \in \mathcal{T}^{n-3}$, and $t \in [0,1]$. In this coordinate system, $Z = (0, 0, 1)$.

Recall that in Brin's construction, $N = \mathcal{D}^2 \times L \subset \mathbb{R}^n$ is an n dimensional manifold, where $L \subset \mathbb{R}^n$ is diffeomorphic to the suspension manifold of the toral automorphism $A : \mathcal{T}^{n-3} \to \mathcal{T}^{n-3}$. So we have $N = \mathcal{D}^2 \times \mathcal{T}^{n-3} \times [0,1]/\sim$, and $\sim$ is the identification $(x, y, 1) \sim (x, Ay, 0)$.

A result of Smale says that any diffeomorphism of $\mathcal{D}^2$ which is the identity restricted to a neighborhood of the boundary $\partial\mathcal{D}^2$ is diffeotopic to the identity map of $\mathcal{D}^2$ (see [S], Theorem B and Theorem 4). So there is a C^∞ map $G : \mathcal{D}^2 \times [0,1] \to \mathcal{D}^2$ such that $G(\cdot, 0) = \mathrm{id}\,|_{\mathcal{D}^2}$ and $G(\cdot, 1) = g$. Further, we require that for each t, $G(\cdot, t) : \mathcal{D}^2 \to \mathcal{D}^2$ is area preserving. This can be done by using similar arguments for Moser theorem. Also, by Corollary 4.2, g is a time one map of some flow g^t on a neighborhood U is $\partial\mathcal{D}^2$. So we may assume further that restricted to U, $G(x, t) = (g^t x, t)$ for any $t \in [0,1]$.

Let $F : K \to N$ be given by $F(x, y, t) = (G(x, t), y, t)$. Clearly F is a diffeomorphism. Take a vector field $Y = DF(Z)$ on N. If we choose local coordinates in N in the same way as we did for K, then $Y(x, y, t) = (Y_1(x, t), 0, 1)$, where $Y_1(x, t) = \dfrac{\partial}{\partial t} G(x, t)$ for any $t \in [0,1]$.

Now we need modify Y such that it vanishes on the boundary $\partial N = \partial\mathcal{D}^2 \times L$. We take a admissible function $\alpha \in C^\infty_\rho(\mathcal{D}^2)$ with $0 \leq \alpha \leq 1$ and $\alpha|_{\mathcal{D}^2 \setminus U} = 1$, and then let

$$X(x, y, t) = (Y_1(x, t), 0, \alpha(x)).$$

Note that for each t, $G(\cdot, t) : \mathcal{D}^2 \to \mathcal{D}^2$ is area preserving. So we have $\operatorname{div} Y_1 = 0$ for every t. Hence, $\operatorname{div} X = 0$ because $\alpha(x)$ is independent of t. Therefore, the flow generated by X is volume preserving. The facts $G(x, t) = (g^t x, t)$ and $\alpha \in C^\infty_\rho(\mathcal{D}^2)$ guarantee that when we embed the system in a given manifold M by the methods discussed in Section 3, the resulting vector field is still smooth.

It is easy to see that the time t map f^t for any t is hyperbolic, because Df^t is expanding on $E^u(g) \oplus E^u(A)$, and contracting on $E^s(g) \oplus E^s(A)$. Therefore, f^t is partially hyperbolic. Also, it is clear that it is center bunched and dynamically coherent. Further, we can see that the system has essential accessibility property. So by Theorem 2.5, f^t is ergodic for any t.

To get Bernoullicity, we can first show that for any $t \neq 0$, f^t has trivial Pinsker algebra. So f^t has the Kolmogorov property. Then we use Theorem 2.8.

If $\dim M = 3$ and 4, we can use the suspension flow of a ergodic hyperbolic diffeomorphism of manifold of dimension 2 or 3, then follow the same method. □

8. Diffeomorphisms with infinitely many ergodic components

In the above sections we introduced how to construct volume preserving hyperbolic ergodic diffeomorphisms and flows on given manifold. These give a complete answer for the case $m = 1$ in our question asked in the Introduction. Now we introduce an example of diffeomorphisms which implies that $m = \infty$ is possible. The construction is due to D. Dolgopyat, H. Hu, and Ya. Pesin (see [DHP]).

Theorem 8.1. *There exists a C^∞ volume preserving hyperbolic diffeomorphism f of $\mathcal{T}^3$ with countably many ergodic components $\{\Lambda_i\}_{i=1}^\infty$ such that each Λ_i is open* (mod 0) *and $f|_{\Lambda_i}$ is Bernoulli.*

Proof. Take a linear automorphism $A : \mathcal{T}^2 \to \mathcal{T}^2$ that has at least 2 fixed points p and p'. Take a direct product $(A, \mathrm{id}) : \mathcal{T}^2 \times \mathcal{S}^1 \to \mathcal{T}^2 \times \mathcal{S}^1$. Divide $\mathcal{S}^1$ into countably many subintervals $\{I_i\}$. For each i we perturb the map $(A, \mathrm{id}\,|_{I_i}) : \mathcal{T}^2 \times I_i \to \mathcal{T}^2 \times I_i$ to get hyperbolicity and ergodicity. The size of perturbations can be chosen in such a way that the resulting map is still C^∞.

Without loss generality we may regard $I_i = I = [0, 1]$. So we need perturb the map $F := (A, \mathrm{id}) : N \to N$, where $N = \mathcal{T}^2 \times I$, by perturbations of arbitrarily small size to get a C^∞ volume preserving hyperbolic Bernoulli diffeomorphism $g : N \to N$ with $D^k g|_{\mathcal{T}^2 \times \{z\}} = D^k F|_{\mathcal{T}^2 \times \{z\}}$ for all $0 \le k < \infty$ and for $z = 0, 1$.

First, we construct a perturbation to get ergodicity. Choose $\epsilon_0 > 0$ small. Then take suitable ϵ, ϵ' such that $0 < \epsilon' < \epsilon < \epsilon_0$. Set $\Omega_1 = B(p, \epsilon_0) \times I$ and consider the coordinate system in Ω_1 originated at $(p, 0)$ with x, y, and z-axes to be unstable, stable, and neutral directions respectively. So if a $w = (x, y, z) \in \Omega_1$ and $F(w) \in \Omega_1$, then $F(w) = (\eta x, \eta^{-1} y, z)$, where η is the larger eigenvalue of A. We choose a C^∞ function $\xi : I \to \mathbb{R}^+$ such that $\xi'(z) > 0$ on $(0, 1)$; $\xi^{(i)}(0) = \xi^{(i)}(1) = 0$ for $i = 0, 1, 2, \dots$; $\|\xi\|_{C^k}$ is small for some $k \ge 0$. Then we choose C^∞ functions $\phi, \psi : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ such that ϕ and ψ are positive constants on $(-\epsilon', \epsilon')$, and zero outside $(-\epsilon, \epsilon)$; $\|\phi\|_{C^k}$, $\|\psi\|_{C^k}$ are small for some $k \ge 0$; $\psi(y) \ge 0$ for any y and $\int_0^{\pm\epsilon} \phi(s) ds = 0$. Now we define the vector field X on Ω_1 by

$$X(x, y, z) = \Big(-\psi(y) \xi'(z) \int_0^x \phi(s) ds, \quad 0, \quad \psi(y) \xi(z) \phi(x) \Big).$$

Clearly X is a divergence free vector field supported on $(-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times I$. We define the perturbation $h_t^{(1)}$ on Ω_1 to be the time t map of the flow generated by X and we set $h_t^{(1)} = \mathrm{id}$ on the complement of Ω_1.

Next we construct a perturbation to remove the zero Lyapunov exponent. Take $q \in \mathcal{T}^2$. Set $\Omega_2 = B((q, 1/2), \epsilon_0)$. Consider the coordinate system in Ω_2 originated at $(q, 1/2)$ with x, y, and z-axes to be unstable, stable, and neutral directions respectively. Choose a C^∞ function $\rho : (-\epsilon_0, \epsilon_0) \to \mathbb{R}^+$ such that $\rho(r) > 0$ if $r < \epsilon'$ and $\rho(r) = 0$ if $r \ge \epsilon$; and $\|\rho\|_{C^k}$ is small. Let $\psi(y)$ be the same function as above. We define the map $h_\tau^{(2)}$ on Ω_2 by

$$h_\tau^{(2)}(x, y, z) = (r \cos \sigma, y, r \sin \sigma),$$

where $r = \sqrt{x^2 + y^2}$, $\sigma = \theta + \tau \psi(y) \rho(r)$, and $\theta = \tan^{-1}(z/x)$. Then we extend it to N by letting $h_\tau^{(2)} = \mathrm{id}$ on $M \backslash \Omega_2$. Clearly $h_\tau^{(2)}$ is a C^∞ volume preserving diffeomorphism for every τ.

Now we set $g = g_{t\tau} = h_t^{(1)} \circ F \circ h_\tau^{(2)}$. g is volume preserving because both $h_t^{(t)}$ and $h_\tau^{(2)}$ are. Next we show that for all $\tau \ge 0$, $t > 0$, $g_{t\tau}$ is ergodic. By Theorem 2.6, g is dynamically coherent. Since g is partially hyperbolic and center bunched, by Theorem 2.5 we only need show that g is essentially accessibe for any $t > 0$.

Since A has the accessibility property, any point $(w,z) \in N$ is accessible to some point in $I_p := \{p\} \times I$ through strong stable and unstable leaves of g. Since accessibility is a reflective and transitive relation, and $g^{-n}(p,z) \to (p,0)$ as $n \to 0$ for any $z \in (0,1)$, we only need show that for any $z_0 \in (0,1)$, there is $z' < z_0$ such that (p,z_0) and (p,z'') are accessible for all $z'' \in [z', z_0]$.

Recall that p' is also a fixed point of A. Let $V^u((p,z_0),g)$ be the local unstable manifold of the map $g = g_{t\tau}$ at (p,z_0), etc. Choose $q_1 \in A^{-n_1}(V^s(p',A)) \cap V^u(p,A)$ and $q_2 \in A^{n_2}(V^u(p',A)) \cap V^s(p,A)$ for some $n_1, n_2 > 0$. Consider a path from (p,z_0) to (q_1,z_1) through $V^u((p,z_0),g)$, then to (p',z_2) through $g^{-n_1}V^s(g^{n_1}(q_1,z_1),g) = g^{-n_1}V^s((p',z_2),g)$, then to (q_2,z_3) through $g^{n_2}V^u((p',z_2),g)$, and then to (p,z_4) through $V^s((q_2,z_3),g)$. Since both $g^{-n_1}V^s((p',z),g)$ and $g^{n_2}V^u((p',z),g))$ are unperturbed, $z_1 = z_2 = z_3$. Observe $g^{-n}(p,z) \to (p,0)$ as $n \to \infty$ for all $z \in (0,1)$. So along $V^u((p,z_0),g)$, the z-coordinates of points are decreasing. We get $z_1 < z_0$. Similar reasons give $z_4 < z_3$. So $z_4 < z_0$ and (p,z_0) and (p,z_4) are accessible. Since the strong stable and unstable curves change smoothly with the point, by shrinking this loop we see that (p,z_0) is accessible to (p,z'') for any $z'' \in [z_4,z_0]$. This is what we need for accessibility property and therefore ergodicity.

Next, we show that there is $\tau > 0$ such that for all small $t > 0$, $g_{t\tau}$ has no zero Lyapunov exponent almost everywhere. Note that if g is ergodic, then the three Lyapunov exponents of g are equal to $\int \log \|Dg|_{E^v(g)}\| d\mu$, where $v = u,c,s$. Also, $\int \log \|Dg|_{E^u(g)}\| d\mu + \int \log \|\det Dg|_{E^c(g)}\| d\mu + \int \log \|\det Dg|_{E^s(g)}\| d\mu = 0$ for any t and τ. Since both $h_t^{(1)}$ and $h_\tau^{(2)}$ preserve the $\mathcal{F}^{uc}$ foliation, the negative Lyapunov exponent of $g_{t\tau}$ does not change with t and τ, though E^s does in general. So if we can show that

$$\int \log \|Dg_{0\tau}|_{E^u(g_{0\tau})}\| d\mu < \int \log \|Dg_{00}|_{E^u(g_{00})}\| d\mu = \log \eta, \tag{8.1}$$

then by continuity of strong unstable manifold under perturbation, we still have $\int \log \|Dg_{t\tau}|_{E^u(g_{t\tau})}\| d\mu < \log \eta$ for small $t > 0$. This gives $\int \log \|Dg_{t\tau}|_{E^c(g_{t\tau})}\| d\mu > 0$. Hence $\lambda^c(g_{t\tau}) > 0$ and $g_{t\tau}$ is hyperbolic.

The arguments to prove (8.1) was used in [SW]. For any $w \in N$, we use the coordinate system in T_wN associated with the splitting $E_F^u(w) \oplus E_F^s(w) \oplus E_F^c(w)$. Given $\tau \geq 0$ and $w \in M$, there exists a unique number $\alpha_\tau(w)$ such that the vector $v_\tau(w) = (1, 0, \alpha_\tau(w))^\perp$ lies in $E_{0\tau}^u(w)$, (where $\perp$ denotes the transpose). By the definition of the function $\alpha_\tau(w)$, we have

$$Dg_{0\tau}(w) v_\tau(w) = \big(\kappa_\tau(w), 0, \kappa_\tau(w)\alpha_\tau(g_{0\tau}(w))\big)^\perp \tag{8.2}$$

for some $\kappa_\tau(w) > 1$. The product $\kappa_\tau(w)\kappa_\tau(g_{0\tau}w) \cdots \kappa_\tau(g_{0\tau}^{n-1}w)$ is roughly the expanding rate of $Dg_{0\tau}^n(w)$ in the unstable direction. Since μ is an invariant measure,

$$L_\tau := \int \log \kappa_\tau(w)\, d\mu = \int \log \|Dg_{0\tau}|_{E^u(g_{0\tau})}\| d\mu. \tag{8.3}$$

On the other hand, restricted to the E^{uc} bundle, the matrix form of the differential $Dh_\tau^{(2)}|_{E^u(w,F) \oplus E^c(w,F)}$ is

$$Dh_\tau^{(2)}(w) = \begin{pmatrix} A(\tau,w), & B(\tau,w) \\ C(\tau,w), & D(\tau,w) \end{pmatrix} = \begin{pmatrix} r_x \cos\sigma - r\sigma_x \sin\sigma, & r_y \cos\sigma - r\sigma_y \sin\sigma \\ r_x \sin\sigma + r\sigma_x \cos\sigma, & r_y \sin\sigma + r\sigma_y \cos\sigma \end{pmatrix},$$

where $r_x = \partial r / \partial x$, etc.

Since $g_{0\tau} = F \circ h_\tau^{(2)}$, we can write (8.2) as

$$Dg_{0\tau}(w)\begin{pmatrix} 1 \\ \alpha_\tau(w) \end{pmatrix} = \begin{pmatrix} \eta A(\tau,w) + \eta B(\tau,w)\alpha_\tau(w) \\ C(\tau,w) + D(\tau,w)\alpha_\tau(w) \end{pmatrix} = \begin{pmatrix} \kappa_\tau(w) \\ \kappa_\tau(w)\,\alpha_\tau(g_{0\tau}(w)) \end{pmatrix},$$

Since $h_\tau^{(2)}$ is volume preserving, $AD - BC = 1$ and therefore, $A + B\alpha = 1/D + (B/D)(C + D\alpha)$. Comparing the components in the equation, we obtain

$$\kappa_\tau(w) = \eta\Big(\frac{1}{D(\tau,w)} + \frac{B(\tau,w)}{D(\tau,w)}\kappa_\tau(w)\,\alpha_\tau(g_{0\tau}(w))\Big).$$

Solving for $\kappa_\tau(w)$ and substituting it in (8.3), we get

$$L_\tau = \log\eta - \int_M \log\big(D(\tau,w) - \eta B(\tau,w)\alpha_\tau(g_{0\tau}(w))\big)dw.$$

By differentiating the equality with respect to τ. we get

$$\begin{aligned} \frac{dL_\tau}{d\tau}\Big|_{\tau=0} &= \int_{\Omega_2} D'_\tau\,dw, \\ \frac{d^2L_\tau}{d\tau^2}\Big|_{\tau=0} &= \int_{\Omega_2}\big[(D'_\tau)^2 - D''_{\tau\tau} + 2\eta B'_\tau\frac{\partial\alpha_\tau(w)}{\partial\tau}(g_{0\tau}(w))\big]_{\tau=0}\,dw. \end{aligned}$$

It can be proved that the first derivative is zero, and the second one is negative. (It take some work to estimate the second derivative.) Therefore, if τ is small, then $L_\tau < L_0 = \log\eta$. This is (8.1).

Bernoullicity follows from hyperbolicity and ergodicity of $g_{t\tau}^n$ for all $n > 0$. $\square$

Further, diffeomorphisms with such properties can be made arbitrarily close to the identity map in C^1 topology. The examples are constructed by H. Hu and A. Taliskaya ([HT]).

Theorem 8.2. *There exists a C^∞ volume preserving hyperbolic diffeomorphism f of a four dimensional manifold M that is arbitrarily close to the identity map in C^1 topology and has countably many open* (mod 0) *ergodic components.*

Proof. Let $g^t : K \to K$ be a geodesic flow on a compact surface of constant negative curvature. Let $G = g^\delta$ be the time δ map of g^t for a small δ such that G has a periodic point p. Let $M = K \times \mathcal{S}^1$ and $F = G \times \mathrm{id} : M \to M$. Clearly $\|F - \mathrm{id}\,\| \le \delta$.

Note that M is a four dimensional manifold with stable, unstable, flow and circle directions E^s, E^u, E^c, and E^n respectively. We perturb F to get the required map.

Partition $\mathcal{S}^1$ into countably many intervals $\{I_n\}$. Let $N = N_n = K \times I_n$ and $S = F|_{N_n}$. Without loss generality we may assume $I_n = [0,1]$. We perturb S on each N_n by three perturbations of arbitrarily small size. The first perturbation is to get ergodicity, and the second is to remove the zero Lyapunov exponents in E^n direction. They are similar to $h_t^{(1)}$ and $h_\tau^{(2)}$ in the construction in the above theorem. The proof of accessibility is more delicate since we need use one dimensional leaves in $\mathcal{F}^u$ and $\mathcal{F}^s$ to access the line $\{p\} \times I$ in a four dimensional space. Since $\mathcal{F}^{un}$ and $\mathcal{F}^{unc}$ foliations are preserved, $\lambda^u + \lambda^n = \int \log|\det DS|_{E^{un}}|d\mu$ and $\lambda^u + \lambda^n + \lambda^c = \int \log|\det DS|_{E^{unc}}|d\mu$ remains the same. So $\lambda^s < 0$ and $\lambda^c = 0$ are unchanged, while λ^n becomes positive.

The last perturbation is to remove the second zero Lyapunov exponent. It is similar to that used for the case $\dim M = 4$ in the proof of Theorem 6.1. $\square$

The diffeomorphisms constructed in the above theorem are C^∞ but close to the identity only in C^1 topology because of the last perturbation is close to the identity only in C^1 topology.

QUESTION. *Can the diffeomorphisms be constructed to be close to the identity map in C^k $(2 \le k \le \infty)$ topology?*

It is easy to believe that the second zero Lyapunov exponent can be removed by a single rotation. However, the same proof just tells us that λ^u decreases. We do not know whether the second zero exponent is removed, though we believe this is the case.

The example is constructed in a four dimensional manifold. We may ask:

QUESTION. *Can a similar example be constructed on a two or three dimensional smooth compact manifold?*

9. INFINITELY MANY ERGODIC COMPONENTS ON ANY GIVEN MANIFOLD

It seems that the result for diffeomorphisms with countably many ergodic components can be generalized to any compact manifold M with $\dim M \ge 3$. We propose the following theorem.

Theorem 9.1. *For every smooth compact manifold M of dimensional $n \ge 3$ there is a volume preserving hyperbolic diffeomorphism that has infinitely many ergodic components which are open* $\pmod 0$.

This can be done by combine the ideas from the previous constructions.

For example, if $\dim M = 3$, we take Katok's diffeomorphism crossing the identity map on $\mathcal{S}^1$ to get a diffeomorphism $F := (g, \mathrm{id}) : N \to N$, where $N = \mathcal{D}^2 \times \mathcal{S}^1$, then use the same perturbations as we did in the construction in Theorem 8.1. Since $F|_{\partial N} = \mathrm{id}$, the system can be embedded in any three dimensional manifold according to the discussion in Section 3.

For the case $\dim M \ge 6$, we need start with the map $F := g \times h \times \mathrm{id} : N \to N$, where $N = \mathcal{D}^2 \times L \times \mathcal{S}^1$, $g : \mathcal{D}^2 \to \mathcal{D}^2$ and $h : L \to L$ are the maps constructed by Katok and Brin respectively. F is ergodic on each $\mathcal{D}^2 \times L \times \{s\}$, and has two zero Lyapunov exponents. We partition $\mathcal{S}^1$ into countably many subintervals $\{I_n\}$, and then use three perturbations on each component $\mathcal{D}^2 \times L \times I_n$ to get ergodicity and to remove zero Lyapunov exponents.

Obviously, the construction does not work if $\dim M = 2$.

QUESTION. *Is there any area preserving hyperbolic diffeomorphism on a smooth compact surface that has infinitely many ergodic components?*

Also, we can ask whether the diffeomorphisms can be made to be arbitrarily close to identity, and whether every compact manifold carries a volume preserving hyperbolic flow with countably many ergodic components.

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