DECAY OF CORRELATIONS FOR PIECEWISE SMOOTH MAPS WITH INDIFFERENT FIXED POINTS

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Abstract. We consider a piecewise smooth expanding map f on the unit interval that has the form $f(x) = x + x^{1+\gamma} + o(x^{1+\gamma})$ near 0, where $0 < \gamma < 1$. We prove that the density function h of an absolutely continuous invariant probability measure μ has order $x^{-\gamma}$ as $x \to 0$, and that the decay rate of correlations with respect to μ is polynomial for Lipschitz functions. Perron-Frobenius operators are the main tool used for proofs.

0. Introduction

Let $f : I \to I$ be a piecewise smooth map on the unit interval I. It is well known that if f is uniformly expanding, then it admits an absolutely continuous invariant probability measure μ, and (f, μ) has exponential decay of correlations. If f has indifferent fixed points, then f still admits an absolutely continuous invariant measure μ. In addition, if f is $C^{1+\gamma}$, $0 < \gamma < 1$, then the measure μ is finite (See e.g. [P]). The purpose of this paper is to show that such systems has polynomial decay of correlations.

We assume that f has an indifferent fixed point 0, and $fx = x + x^{1+\gamma} + o(x^{1+\gamma})$ near 0. We use Perron-Frobenius operator $\mathcal{L}$ to get the density function h. The fact $\mathcal{L}h = h$ implies that as $x \to 0$, $h(x)$ goes to infinite just like $x^{-\gamma}$ multiplied by a constant related to the value of f' and h at $f^{-1}(0)$. Then we use $\eta(x) = \frac{h(x)}{h(fx)f'(x)}$, instead of $\frac{1}{f'(x)}$, to define a different operator $\tilde{\mathcal{L}}$. This operator preserves L^1 norms and leaves constant functions invariant. So $\tilde{\mathcal{L}}^n g \to \mu(g)$ for any continuous function g. Moreover, if higher order terms are ignored, then near 0, $\tilde{\mathcal{L}}g(x) \approx (1 - x_1^\gamma)g(x_1) + x_1^\gamma \bar{g}(x_1)$, where x_1 is the preimage of x near 0, and $\bar{g}$ is the average of g with weight η at the rest of preimages (see (4.3) for details). Since restricted to a neighborhood of 0, all backward orbits approach to 0 in a polynomial rate, the rate of the convergence $\tilde{\mathcal{L}}^n g \to \mu(g)$, both in $L^1(I, \mu)$ and in measure, is polynomial. Therefore the rate of decay of correlations is polynomial as well.

We state assumptions and the main results in §1. Theorem A, which is concerning existence and properties of density functions of invariant measures, is proved in §3. Theorem B and its corollary, which deal with decay rate of correlations of Lipschitz functions and mixing rate of sets respectively, are proved in §7. To obtain Theorem

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B, we prove Proposition 5.2 in §5 and §6, which asserts that the rate of convergence $\tilde{\mathcal{L}}^n g \to \mu(g)$ is polynomial.

1. Assumptions, Statements of Results and Notations

Let $I = [0, 1]$ be the unit interval and $f : I \to I$ be a piecewise smooth map. A fixed point p of f is called *indifferent* if $fp = p$ and $\lim_{x\to p} f'(x) = 1$.

Assumptions. *Let $f : I \to I$ such that*

(I) *There is a finite partition $\xi = \{I_0, I_1, \cdots, I_Q\}$ into subintervals such that for each q, restricted to I_q, $f|_{\text{int}\, I_q}$ is twice differentiable and $f|_{\text{int}\, I_q}$ maps $\text{int}\, I_q$ to $(0, 1)$ diffeomorphically.*
(II) *0 is an indifferent fixed point of f.*
(III) *$f' > 1$ on $(0, 1]$, and f'' is bounded on $[\tau, 1]$ $\forall \tau > 0$.*

Moreover, we need the following assumption for technical reasons.

(IV) *Near $x = 0$, f and its derivative have the form*

$$f(x) = x + x^{1+\gamma} + x^{1+\gamma}\delta_0(x), \tag{1.1}$$
$$f'(x) = 1 + (1+\gamma)x^\gamma + x^\gamma \delta_1(x), \tag{1.2}$$
$$f''(x) = \frac{\gamma(1+\gamma) + \delta_2(x)}{x^{1-\gamma}}, \tag{1.3}$$

where $\delta_i(x) \to 0$ as $x \to 0$ for $i = 0, 1, 2$.

The last assumption says that f is equal to $x + x^{1+\gamma}$ plus higher order terms, and the first and the second derivative of the higher order terms are still of higher orders.

We denote by I_0 the element of the partition ξ that contains 0.

Theorem A. *Suppose $f : I \to I$ satisfies Assumption (I)–(IV). Then f has an absolutely continuous invariant probability measure μ whose density function $h(x)$ satisfies*

i) $0 < h(x) < \infty$ $\forall x \in (0, 1]$*;*
ii) *h is Lipschitz on* $[\tau, 1]$ $\forall 0 < \tau < 1$*;*
iii) $\exists R > 0$ *such that*

$$|x^\gamma h(x) - \sigma_0| \le R \max\{x^\gamma, \delta_1(x)\},$$

where $\sigma_0 = \lim_{x\to 0} \sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \frac{h(\bar{x}_1)}{f'(\bar{x}_1)}$ *is a constant. In particular*

$$\lim_{x\to 0} x^\gamma h(x) = \sigma_0.$$

The part of existence of absolutely continuous invariant measures was proved by Pianigiani ([P]) in more general setting by using the first return map. For Part iii), a similar result can be seen in [CF1] and [CF2] for a map with the form $fx = \frac{x}{1-x}$ as $0 \le x \le \frac{1}{2}$, which admits σ-finite absolutely continuous invariant measure.

For a Lipschitz function F, denote by $\|F\|$ the C^0 norm.

Theorem B. *Suppose $f : I \to I$ satisfies Assumption (I)–(IV). Let μ be the absolutely continuous invariant probability measure and let $\beta = \gamma^{-1}$. Then*

i) *for any Lipschitz function G, there is a constant $C = C(G) > 0$ such that for any Lipschitz function F,*

$$\Big|\int (F \circ f^n) G d\mu - \int F d\mu \int G d\mu\Big| \le \frac{C}{n^{\beta-1}} \|F\| \qquad \forall n > 0;$$

ii) *there exist Lipschitz functions G and F, and a constant $C' > 0$ such that*

$$\Big|\int (F \circ f^n) G d\mu - \int F d\mu \int G d\mu\Big| \ge \frac{C'}{n^{\beta-1}} \qquad \forall n > 0.$$

A result similar to Part i) has been proved by L.-S. Young recently in more general setting (see [Y]). However, her method is quite different with ours. She uses tails of tower, and we use Perron-Frobenius operators. Earlier, M. Mori proved polynomial decay of correlations for piecewise linear maps (see [M]).

Remark. By the proof of the theorem, we can see that Part i) still holds if we use $L^\infty(I,\mu)$ function F and the $L^\infty(I,\mu)$ norm $\|F\|_\infty$ instead of Lipschitz function and C^0 norm respectively. On the other hand, we can find C^∞ functions F and G satisfying the inequality in Part ii).

Denote $\xi_m = \bigwedge_{i=0}^{m-1} f^{-i}\xi$. So if $E \in \xi_m$, then $E = \bigcap_{i=0}^{m-1} f^{-i} I_{q_i}$ for some $q_0, \cdots, q_{m-1}$. Also we denote by $E^{(m)}$ the element of ξ_m containing 0.

Corollary. *Under the supposition of Theorem B, there exist constants $C > C' > 0$ and $l > 0$ such that for any $m \ge 0$, $E \in \xi_m$, and for any measurable set $E' \subset [0,1]$,*

$$\big|\mu(f^{-n-m}E' \cap E) - \mu E \cdot \mu E'\big| \le \frac{C m^{\beta-1}}{(n-l)^{\beta-1}} \mu E \cdot \mu E' \qquad \forall n > l,$$

and if in addition $m \ge l$ and $E = E^{(m)}$, then

$$\big|\mu(f^{-n-m}E' \cap E) - \mu E \cdot \mu E'\big| \ge \frac{C' m^{\beta-1}}{(n+m)^{\beta-1}} \mu E \cdot \mu E' \qquad \forall n > 0.$$

We introduce some notations.

Let I_0 be the element of the partition ξ containing 0. For $x \in I_0$, we denote $x_0 = x$ and $x_{i+1} = f^{-1}x_i \cap I_0$ $\forall i > 0$. Choose a small neighborhood $P_0 \subset I_0$ of 0.

For any function g, if $x \in I_0$, then we denote

$$\sigma_g(x) = \sum_{\bar{x}_1 \in f^{-1}(fx) \setminus I_0} \frac{g(\bar{x}_1)}{f'(\bar{x}_1)}. \tag{1.4}$$

We should note that $\sigma_g(x)$ depends on values of g at $f^{-1}(fx)$ but at x itself.

Take nondecreasing functions $\rho_\pm(x) \ge 0$ and denote $B(x, \rho(x)) = \big(x - \rho_-(x), x + \rho_+(x)\big)$. We require that $\rho_\pm(x)$ are chosen in such a way that $\rho_\pm(x) = O(x^{1+\gamma})$ on P_0, and $fB(x, \rho(x)) \supset B(fx, \rho(fx))$ $\forall x \in I$, and $y \in B(x, \rho(x))$ if and only if $x \in B(y, \rho(y))$. The latter implies $\rho_+(x) > \rho_-(x)$ on P_0. So $B(x, \rho(x))$ is not a ball in Euclid metric. Since $\rho_\pm(x)$ are nondecreasing, we have $\rho(x) \ge \bar{\rho}$ for some $\bar{\rho} > 0$ on $I \setminus I_0$.

For any $n \ge 0$, denote $B_n(x, \rho) = \{y \in I : d(f^i y, f^i x) \le \rho(f^i x) \ \forall 0 \le i \le n\}$.

We always denote $\beta = \gamma^{-1}$. Choose $\beta_- < \beta < \beta_+$ such that $\beta_+ - \beta$ and $\beta - \beta_-$ are small, for example, less than 0.1 and $0.1(\beta - 1)$.

2. Preliminary

Lemma 2.1. *Let $x \in I_0$. For any $\theta \ge 0$,*

$$\frac{x^\theta}{x_1^\theta} \cdot \frac{d(x_1, y_1)}{d(x, y)} = \frac{x^\theta}{x_1^\theta} \cdot \frac{1}{f'(x)} + o(x^\gamma) = 1 - (1 + \gamma - \theta)x^\gamma + o(x^\gamma), \quad x \to 0,$$

where $y \in B(x, \rho(x))$ and $y_1 \in B(x_1, \rho(x_1))$.

Proof. This is because by Assumption (IV), $x = fx_1 = x_1(1 + x_1^\gamma + o(x^\gamma)$, and $d(x, y) = \big(f'(x_1) + o(x^\gamma)\big)d(x_1, y_1) = \big(1 + (1 + \gamma)x^\gamma + o(x^\gamma)\big)d(x_1, y_1)$. $\square$

Lemma 2.2. *Given $\beta_- < \beta < \beta_+$, we can choose P_0 small enough such that for any $x \in P_0$,*

i) *if $x = x_0 \ge \Big(\dfrac{\beta_-}{r}\Big)^\beta$ for some $r > 0$, then $x_n \ge \Big(\dfrac{\beta_-}{r+n}\Big)^\beta$;*

ii) *if $x = x_0 \le \Big(\dfrac{\beta_+}{r}\Big)^\beta$ for some $r > 0$, then $x_n \le \Big(\dfrac{\beta_+}{r+n}\Big)^\beta$.*

Proof. If x is small, then we can find $1 < \lambda < \beta/\beta_-$ such that $f(x) \le x(1 + \lambda x^\gamma)$.

Suppose $x \le \Big(\dfrac{\beta_-}{r}\Big)^\beta$. We have

$$f(x) \le \Big(\frac{\beta_-}{r}\Big)^\beta \Big(1 + \frac{\lambda\beta_-}{r}\Big) = \beta_-^\beta \frac{r + \lambda\beta_-}{r^{\beta+1}}.$$

Note that $\lambda\beta_- < \beta$. If r is large enough, then

$$\Big(1 - \frac{1}{r}\Big)^\beta \Big(1 + \frac{\lambda\beta_-}{r}\Big) \le 1 \qquad \text{or} \qquad \Big(r - 1\Big)^\beta \Big(r + \lambda\beta_-\Big) \le r^{\beta+1}.$$

So we get that

$$f(x) \leq \Big(\frac{\beta_-}{r-1}\Big)^{\beta}.$$

This implies the result in (i).

Part (ii) can be proved similarly. □

Define

$$\Delta(x,y) = \begin{cases} 1 + \frac{J_0}{x} d(x,y), & \forall x \in P_0, y \in B\big(x, \rho(x)\big); \\ 1 + J d(x,y), & \forall x \in I \backslash P_0, y \in B\big(x, \rho(x)\big), \end{cases}$$

where $J, J_0 > 0$ are constants satisfying the proposition below.

Proposition 2.3. *(Distortion Estimates) There exist constants $J, J_0 > 0$ such that for all $x \in I$, $y \in B\big(x, \rho(x)\big)$,*

i) *if $x_1 \in f^{-1}x$, $y_1 \in f^{-1}y \cap B(x_1, \rho(x_1))$, then*

$$\Delta(x_1, y_1) \cdot \frac{f'(x_1)}{f'(y_1)} \leq \Delta(x,y);$$

ii) *for all $n > 0$, if $x_n \in f^{-n}x$, $y_n \in f^{-n}y \cap B_n(x_n, \rho)$, then*

$$\frac{\big(f^n\big)'(x_n)}{\big(f^n\big)'(y_n)} \leq \Delta(x,y).$$

Proof. i) First we suppose $x \in P_0$. By (1.3) and the fact $f'(y) > 1$, there is a constant $c > 0$ such that

$$\frac{f'(x)}{f'(y)} < 1 + \big(f'(x) - f'(y)\big) \leq 1 + c x^{\gamma} \frac{d(x,y)}{x}.$$

Note that $x^{-1} d(x,y)$ is of order x^{γ}. So by Lemma 2.1 with $\theta = 1$ we have

$$\begin{aligned}
\Delta(x_1, y_1) \cdot \frac{f'(x_1)}{f'(y_1)} &\leq \Big(1 + J_0 \frac{d(x_1,y_1)}{x_1}\Big) \cdot \Big(1 + c x_1^{\gamma} \frac{d(x_1,y_1)}{x_1}\Big) \\
=& 1 + J_0 \big[1 + \frac{c x_1^{\gamma}}{J_0} + O(x_1^{2\gamma})\big] \frac{d(x_1,y_1)}{x_1} \\
=& 1 + J_0 \Big(1 + \frac{c x_1^{\gamma}}{J_0} + O(x_1^{2\gamma})\Big)\Big(1 - \gamma x^{\gamma} + o(x^{\gamma})\Big) \frac{d(x,y)}{x}
\end{aligned}$$

If J_0 is large enough, then the right side is less than $1 + J_0 x^{-1} d(x,y)$.

For the case $x \notin P_0$, the result is clear since f is uniformly expanding outside P_0.

ii) can be obtained from i) by induction. □

Remark. The ratio $\dfrac{\big(f^n\big)'(x_n)}{\big(f^n\big)'(y_n)}$ only depends on preimages of x and y. So if $f^{n-1}x_n \in I \backslash I_0$, then we still have

$$\frac{\big(f^n\big)'(x_n)}{\big(f^n\big)'(y_n)} \leq 1 + J d(x,y)$$

for some $J > 0$ even if $x \in P_0$.

Recall the definition (1.4) of σ_g.

Corollary 2.4. *Let $x, y \in P_0$. If g satisfies $g(\bar{y}_1) \le g(\bar{x}_1)\Delta(\bar{x}_1, \bar{y}_1)$ for all $\bar{x}_1 \in f^{-1}x \backslash I_0$, $\bar{y}_1 \in f^{-1}y \cap B(\bar{x}_1, \bar{\rho})$, then*

$$\sigma_g(y_1) \le \sigma_g(x_1)\big(1 + Jd(x, y)\big).$$

Proof. By (1.4) and Proposition 2.3.i),

$$\frac{\sigma_g(y_1)}{\sigma_g(x_1)} = \frac{\sum\limits_{\bar{y}_1 \in f^{-1}y\backslash I_0} g(\bar{y}_1)/f'(\bar{y}_1)}{\sum\limits_{\bar{x}_1 \in f^{-1}x\backslash I_0} g(\bar{x}_1)/f'(\bar{x}_1)} \le \max\Big\{\frac{g(\bar{y}_1)}{g(\bar{x}_1)} \cdot \frac{f'(\bar{x}_1)}{f'(\bar{y}_1)}\Big\} \le \max\Big\{\Delta(\bar{x}_1, \bar{y}_1)\frac{f'(\bar{x}_1)}{f'(\bar{y}_1)}\Big\}$$

where max is taken over all pairs $\bar{x}_1 \in f^{-1}x\backslash I_0$ and $\bar{y}_1 \in f^{-1}y \cap B(\bar{x}_1, \bar{\rho})$. Since $\bar{x}_1, \bar{y}_1 \notin I_0$, $\Delta(\bar{x}_1, \bar{y}_1)\dfrac{f'(\bar{y}_1)}{f'(\bar{x}_1)} \le \big(1 + Jd(\bar{x}_1, \bar{y}_1)\big)\dfrac{f'(\bar{y}_1)}{f'(\bar{x}_1)} \le 1 + Jd(x, y)$. □

3. The Density Function

In this section we prove Theorem A, and then prove a result (Lemma 3.5) which implies that decreasing rate of h is arbitrarily large as x goes to 0.

Proof of Theorem A.

Define Perron-Frobenius Operator $\mathcal{L} = \mathcal{L}_{-\log f'}$ from the set of continuous functions on $(0, 1]$ to itself by

$$\mathcal{L}g(x) = \sum_{\hat{x}_1 \in f^{-1}x} \frac{g(\hat{x}_1)}{f'(\hat{x}_1)}.$$

Let v denote the Lebesgue measure on I. Clearly, $v(\mathcal{L}g) = v(g)$ for any integrable function g on $(0, 1]$.

Also it is well known that for any fixed point h of $\mathcal{L}$, a measure μ given by $\mu(g) = v(g \cdot h)$ is an invariant measure of f. In fact, we can check directly that $\mathcal{L}\big(h \cdot (g \circ f)\big) = g \cdot (\mathcal{L}h)$, then we have $\mu(g \circ f) = v\big(h \cdot (g \circ f)\big) = v\Big(\mathcal{L}\big(h \cdot (g \circ f)\big)\Big) = v\big((\mathcal{L}h) \cdot g\big) = v(h \cdot g) = \mu(g)$. (See e.g. [B] for more details.)

Let $\mathcal{B}$ denote the set of continuous functions g on $(0, 1]$ with the norm

$$\|g\| = \sup_{x \in (0,1]} \big\{xg(x)\big\}. \tag{3.1}$$

It is easy to check that $\mathcal{B}$ is a Banach space and $\mathcal{L}$ is a Linear operator on $\mathcal{B}$. Lemma 3.1 below implies that the operator $\mathcal{L}$ is continuous.

Put

$$\mathcal{G} = \{g \in \mathcal{B} : g > 0,\ v(g) = 1,\ g(y) \le g(x)\Delta(x, y)\ \forall x \in I, y \in B(x, \rho(x)),$$
$$x^\gamma g(x) \le H_0\ \forall x \in P_0\}.$$

where H_0 is a constant to be determined later.

$\mathcal{G}$ is not empty since $(1 - \gamma)x^{-\gamma} \in \mathcal{G}$. It is clear that $\mathcal{G}$ is a convex set. By Lemma 3.2 and 3.3, $\mathcal{G}$ is compact and $\mathcal{L}\mathcal{G} \subset \mathcal{G}$ if H_0 is large enough. So by Schauder-Tychonoff fixed point theorem (see e.g. [DS]), $\mathcal{L}$ has a fixed point $h \in \mathcal{G}$, and therefore, i) and ii) follows from the definition of $\mathcal{G}$. Part iii) can be obtained from Lemma 3.4 and the fact $\phi(x) = (1 + \gamma)x^\gamma + o(x^\gamma)$. □

Lemma 3.1. *$\mathcal{L}$ is a bounded linear operator.*

Proof. Since $\mathcal{L}$ is a positive operator and x^{-1} is the maximal element in the unit ball with respect to the norm in (3.1), we only need to prove that $x\mathcal{L}(x^{-1})$ is bounded.

Note $f'(x) \geq 1$. We have

$$\mathcal{L}(\frac{1}{x}) < \sum_{\hat{x}_1 \in f^{-1}x} \frac{1}{\hat{x}_1} \leq \frac{1}{x_1} + \sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \frac{1}{\bar{x}_1} \leq \frac{1}{x} \max_{z \in [0,1]} \{f'(z)\} + \sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \frac{1}{\bar{x}_1},$$

where the last inequality follows from the fact $x = f(x_1) \leq x_1 \max_{z \in [0,1]} \{f'(z)\}$. Since the second term is bounded, $\|\mathcal{L}\| = \sup_{x \in (0,1]} \{x\mathcal{L}(x^{-1})\}$ is finite. □

Lemma 3.2. *The set $\mathcal{G}$ is compact.*

Proof. First, $\mathcal{G}$ is a bounded set. In fact, for any $g \in \mathcal{G}$, if $x \notin P_0$, then

$$1 \geq \int_{B(x,\rho(x))} g(y)dy \geq g(x) \frac{1}{1 + J\rho(x)} \cdot 2\rho(x).$$

That is,

$$xg(x) \leq \frac{x}{2\rho(x)} \big(1 + J\rho(x)\big) \leq \sup_{x \notin P_0} \Big\{ \frac{x\big(1 + J\rho(x)\big)}{2\rho(x)} \Big\}.$$

If $x \in P_0$, then $xg(x) \leq H_0 x^{1-\gamma} \leq H_0$.

Using the facts that $g(y) \leq \Delta(x,y)g(x)$ $\forall y \in B(x,\rho(x))$ and $xg(x) \leq H_0 x^{1-\gamma}$ $\forall x \in P_0$, we know that $\mathcal{G}$ is also an equicontinuous set. □

Lemma 3.3. *If H_0 is large enough, then $\mathcal{L}\mathcal{G} \subset \mathcal{G}$.*

Proof. Take $g \in \mathcal{G}$. We prove $\mathcal{L}g \in \mathcal{G}$.

It is clear that $\mathcal{L}g > 0$ and $\upsilon(\mathcal{L}g) = \upsilon(g) = 1$.

If $x, y \in I$ with $d(x,y) \leq \rho(x)$, then Proposition 2.3.i) and the same arguments as in the proof of Corollary 2.4 give

$$\frac{\mathcal{L}g(y)}{\mathcal{L}g(x)} = \frac{\sum_{\hat{y}_1 \in f^{-1}y} g(\hat{y}_1)/f'(\hat{y}_1)}{\sum_{\hat{x}_1 \in f^{-1}x} g(\hat{x}_1)/f'(\hat{x}_1)} \leq \max \Big\{ \Delta(\hat{x}_1, \hat{y}_1) \frac{f'(\hat{y}_1)}{f'(\hat{x}_1)} \Big\} \leq \Delta(x,y),$$

where max is taken over all pairs $\hat{x}_1 \in f^{-1}x$ and $\hat{y}_1 \in f^{-1}y \cap B(\hat{x}_1, \rho(\hat{x}_1))$.

Suppose $x \in P_0$. Using Lemma 2.1 with $\theta = \gamma$ and using the fact $x^\gamma g(x) \leq H_0$ $\forall x \in P_0$, we get

$$\begin{aligned} x^\gamma \mathcal{L}g(x) =& x_1^\gamma g(x_1) \frac{x^\gamma}{x_1^\gamma} \frac{1}{f'(x_1)} + x^\gamma \sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \frac{g(\bar{x}_1)}{f'(\bar{x}_1)} \\ \leq& H_0 \Big[1 - x^\gamma + o(x^\gamma) + \frac{x^\gamma}{H_0} \sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \frac{g(\bar{x}_1)}{f'(\bar{x}_1)} \Big]. \end{aligned}$$

Since all element g in $\mathcal{G}$ are uniformly bounded on $I \setminus I_0$, the summation in the second term are bounded. So if we take H_0 large enough, then the right side of the inequality is less than H_0. □

Lemma 3.4. *There exists $R > 0$ such that*

$$\big|h(x)\phi(x) - (1+\gamma)\sigma_0\big| \le R\max\{x^\gamma, \delta_1(x)\},$$

where $\phi(x) = f'(x) - 1 = (1+\gamma)x^\gamma + x^\gamma\delta_1(x)$.

Proof. Denote $\alpha(x) = \max\{x^\gamma, |\delta_1(x)|\}$ for $x \in P_0$. We may assume that $\alpha(x)$ is nondecreasing on P_0, otherwise we use $\max\limits_{0\le y\le x}\{\alpha(y)\}$ instead.

First we claim that there exist $R > 0$ such that if

$$h(x)\phi(x) \ge \Big(1+\gamma+c+R\alpha(x_1)\Big)\sigma_0$$

for some $c \ge 0$ and $x \in P_0$, then

$$h(x_1)\phi(x_1) \ge \Big(1+\gamma+c\big(1+\frac{1}{2}x_1^\gamma\big)+R\alpha(x_2)\Big)\sigma_0.$$

In fact, since $\mathcal{L}h = h$, we have that for $x \in P_0$,

$$h(x_1) = \big(1+\phi(x_1)\big)\big(h(x) - \sigma_h(x_1)\big) \ge \big(1+\phi(x_1)\big)\big(h(x) - \sigma_0 - J\sigma_0 x_1\big), \tag{3.2}$$

where $\sigma_h(x_1) \le \sigma_0(1+Jx_1)$ follows from Corollary 2.4. Also, it is easy to check by (1.1) and the definition of $\delta_1(x)$ that

$$\big(1+\phi(x_1)\big)\cdot\frac{\phi(x_1)}{\phi(x)} = 1 + x_1^\gamma + x_1^\gamma\delta_1^*(x_1).$$

for some $\delta_1^*(x)$ which is bounded by $\delta_1(x)$ multiplied by a constant coefficient. So by (3.2) we get

$$\begin{aligned}
\frac{\phi(x_1)h(x_1)}{\sigma_0} \ge& \frac{\phi(x)h(x)}{\sigma_0}\cdot\big(1+\phi(x_1)\big)\frac{\phi(x_1)}{\phi(x)} - \big(1+Jx_1)\phi(x_1)\big(1+\phi(x)\big) \\
\ge&\big(1+\gamma+c+R\alpha(x_1)\big)\big(1+x_1^\gamma+x_1^\gamma\delta_1^*(x_1)\big) \\
&\quad - \big((1+\gamma)x_1^\gamma + x_1^\gamma\delta_1(x_1)\big)\big(1+\phi(x_1)\big) - Jx_1\phi(x_1)\big(1+\phi(x_1)\big) \\
=&\big(1+\gamma+c+R\alpha(x_1)\big) + \big(c+R\alpha(x_1)\big)\big(x_1^\gamma+x_1^\gamma\delta_1^*(x_1)\big) + \big(1+\gamma)x_1^\gamma\delta_1^*(x_1) \\
&\quad - (1+\gamma)x_1^\gamma\phi(x_1) - x_1^\gamma\delta_1(x_1)\big(1+\phi(x_1)\big) - Jx_1\phi(x_1)\big(1+\phi(x_1)\big).
\end{aligned}$$

If P_0 is small enough, then $|\delta_1^*(x_1)| \le \dfrac{1}{2}$ and therefore $cx_1^\gamma\big(1+\delta_1^*(x_1)\big) \ge \dfrac{1}{2}cx_1^\gamma$. Note that $\alpha(x)$ is greater than or equal to $\delta_1(x)$ and x^γ. So

$$\frac{1}{2}R\alpha(x_1) + \big(1+\gamma)\delta_1^*(x_1) - (1+\gamma)\phi(x_1) - \big(\delta_1(x_1) + Jx_1^{1-\gamma}\phi(x_1)\big)\big(1+\phi(x_1)\big) > 0.$$

if R is sufficiently large. Hence we have

$$\frac{\phi(x_1)h(x_1)}{\sigma_0} \ge \big(1+\gamma+c+R\alpha(x_1)\big) + \frac{1}{2}cx_1^\gamma \ge \big(1+\gamma+c+R\alpha(x_2)\big) + \frac{1}{2}cx_1^\gamma.$$

It means that the claim is true.

Using this claim we can see that

$$\phi(x)h(x) \le (1+\gamma)\sigma_0 + 2R\sigma_0\alpha(x_1) \qquad \forall x \in P_0.$$

Otherwise we may have

$$\phi(x)h(x) \ge (1+\gamma)\sigma_0 + 2R\sigma_0\alpha(x_1) = (1+\gamma)\sigma_0 + c\sigma_0 + R\sigma_0\alpha(x_1)$$

for some $x \in P_0$, where $c = R\alpha(x_1) > 0$. Then by using the claim repeatedly, and using the fact $c \cdot \Big(1 + \frac{1}{2}\sum_{i=1}^{n-1} x_i^\gamma\Big)\big(1 + \frac{1}{2}x_n^\gamma\big) \ge c \cdot \Big(1 + \frac{1}{2}\sum_{i=1}^{n} x_i^\gamma\Big)$ we get that

$$\phi(x_n)h(x_n) \ge (1+\gamma)\sigma_0 + R\sigma_0\alpha(x_{n+1}) + c\sigma_0 \cdot \Big(1 + \frac{1}{2}\sum_{i=1}^{n} x_i^\gamma\Big).$$

By Lemma 2.2, $x_i^\gamma \ge \dfrac{\beta_-}{r+i}$ $\forall i \ge 0$ for some $r > 0$ and therefore $\sum_{i=1}^{\infty} x_i^\gamma$ diverges. This contradicts to the fact that $x^\gamma h(x)$ is bounded for all $x \in P_0$.

By using $\phi(x)h(x) > 0$, the inequality of the other direction can be proved similarly. $\square$

Lemma 3.5. *For any $\gamma' > 0$, we can choose P_0 small enough such that*

$$h(y) \ge h(x) \cdot \Big(1 + \frac{J_0'}{x^{1-\gamma'}} d(x,y)\Big) \qquad \forall x \in P_0,\ x - \rho(x) \le y \le x$$

for some $J_0' > 0$.

Proof. Denote $\tau = \inf_{x\in(0,1]} \Big\{\dfrac{1-\gamma}{x^\gamma h(x)}\Big\}$. By Lemma 2.1, there exists $c > 0$ such that $\dfrac{x^\gamma}{x_1^\gamma} \cdot \dfrac{1}{f'(x)} = 1 - x^\gamma + o(x^\gamma) > 1 - cx^\gamma$ for all $x \in P_0$. We take $H_0' \le \min_{x\in P_0}\{\dfrac{\tau\sigma_h(x)}{c}\}$ and then define

$$\mathcal{G}_1 = \Big\{g \in \mathcal{G} : g(x) \ge \tau h(x)\ \forall x \in (0,1],\quad x^\gamma g(x) \ge H_0'\ \forall x \in P_0,$$
$$g(y) \ge g(x)\big(1 + \frac{J_0'}{x^{1-\gamma'}} d(x,y)\big)\ \forall x \in P_0,\ x - \rho(x) \le y \le x\Big\}.$$

$\mathcal{G}_1$ is not empty because $(1-\gamma')x^{-\gamma'} \in \mathcal{G}_1$. Clearly, $\mathcal{G}_1$ is compact since it is closed in $\mathcal{G}$. We will prove $\mathcal{L}\mathcal{G}_1 \subset \mathcal{G}_1$. Then we can take h as a fixed point of $\mathcal{L}$ in $\mathcal{G}_1$, and therefore h has the required property.

Let $g \in \mathcal{G}_1$. First, we have

$$\mathcal{L}g(x) \ge \mathcal{L}\tau h(x) = \tau\mathcal{L}h(x) = \tau h(x).$$

Secondly, since $\sigma_g(x) \ge \tau\sigma_h(x)$ and $H_0' \le c^{-1}\tau\sigma_h(x)$ $\forall x \in P_0$, we get

$$x^\gamma \mathcal{L}g(x) = x_1^\gamma g(x_1)\frac{x^\gamma}{x_1^\gamma}\frac{1}{f'(x_1)} + x^\gamma\sigma_g(x_1) \ge H_0'\Big(1 - cx^\gamma + \frac{x^\gamma}{H_0'}\tau\sigma_h(x_1)\Big) \ge H_0'.$$

Now it remains to check $\mathcal{L}g(y) \ge \mathcal{L}g(x)\Big(1 + \frac{J_0'}{x^{1-\gamma'}}d(x,y)\Big)$. That is,

$$\frac{g(y_1)}{f'(y_1)} + \sigma_g(y_1) \ge \Big(\frac{g(x_1)}{f'(x_1)} + \sigma_g(x_1)\Big)\Big(1 + \frac{J_0'}{x^{1-\gamma'}}d(x,y)\Big). \tag{3.3}$$

By (1.3), if $x \in P_0$, then $\frac{f'(x)}{f'(y)} \ge 1 + f'(x) - f'(y) \ge 1 + \frac{c'}{x^{1-\gamma}}d(x,y)$ for some $c' > 0$. Also, using Lemma 2.1 for $\theta = 1 - \gamma'$ we have

$$\begin{aligned}\frac{g(y_1)}{f'(y_1)} \ge& \frac{g(x_1)}{f'(x_1)}\Big(1 + \frac{J_0'}{x_1^{1-\gamma'}}d(x_1,y_1)\Big)\Big(1 + \frac{c'}{x^{1-\gamma}}d(x,y)\Big)\\ \ge& \frac{g(x_1)}{f'(x_1)}\ \Big(1 + J_0'\big(1 - (\gamma+\gamma')x^\gamma + o(x^\gamma)\big)\frac{d(x,y)}{x^{1-\gamma'}} + c'\frac{d(x,y)}{x^{1-\gamma}}\Big)\\ \ge& \frac{g(x_1)}{f'(x_1)}\Big(1 + J_0'\frac{d(x,y)}{x^{1-\gamma'}} + \frac{c'}{2}\frac{d(x,y)}{x^{1-\gamma}}\Big),\end{aligned}$$

if P_0 is small enough. Therefore, using Corollary 3.4 and interchanging the roles of x and y, we can see that (3.3) holds if we show

$$\frac{g(x_1)}{f'(x_1)}\frac{c'}{2}\frac{d(x,y)}{x^{1-\gamma}} \ge J\sigma_g(y_1)d(x,y) + \sigma_g(x_1)\frac{J_0'}{x^{1-\gamma'}}d(x,y),$$

or

$$x^{\gamma-\gamma'}\frac{c'g(x_1)}{2f'(x_1)} \ge Jx^{1-\gamma'}\sigma_g(y_1) + J_0'\sigma_g(x_1).$$

However, this is true if P_0 is small, because $x^{\gamma-\gamma'}g(x) \ge x^{\gamma-\gamma'}x^{-\gamma}H_0' = x^{-\gamma'}H_0' \to \infty$, while all other quantities are bounded as $x \to 0$. $\square$

4. The Operator $\tilde{\mathcal{L}}$

Take $\eta(x) = \frac{h(x)}{f'(x)h(fx)}$ if $x > 0$ and $\eta(0) = 1$. By Lemma 4.4 below, $\eta(x)$ is continuous on each I_q.

Define a new Perron-Frobenius Operator $\tilde{\mathcal{L}} = \mathcal{L}_{\log\eta}$ from the set of continuous functions on $[0,1]$ to itself by

$$\tilde{\mathcal{L}}g(x) = \sum_{\hat{x}_1\in f^{-1}x}\eta(\hat{x}_1)g(\hat{x}_1),$$

or equivalently,

$$\tilde{\mathcal{L}}g(x) = \frac{1}{h}\mathcal{L}(hg) = \frac{1}{h(x)}\sum_{\hat{x}_1\in f^{-1}x}\frac{h(\hat{x}_1)}{f'(\hat{x}_1)}g(\hat{x}_1).$$

Recall that the measure μ, defined by $\mu(g) = \upsilon(hg)$, is an f invariant measure, where υ is the Lebesgue measure on I.

Lemma 4.1. *The operator $\tilde{\mathcal{L}}$ has the following properties.*

i) *$\tilde{\mathcal{L}}c = c$ for any constant function c.*

ii) *$\mu(\tilde{\mathcal{L}}g) = \mu(g)$ for any integrable function g.*

Proof. i) Using $\mathcal{L}h = h$, we get $\tilde{\mathcal{L}}c = \frac{1}{h}\mathcal{L}(ch) = \frac{c}{h}\mathcal{L}h = \frac{c}{h}h = c$.

ii) Since $\upsilon(\mathcal{L}g) = \upsilon(g)$, $\mu(\tilde{\mathcal{L}}g) = \upsilon\big(h \cdot \frac{1}{h}\mathcal{L}(hg)\big) = \upsilon\big(\mathcal{L}(hg)\big) = \upsilon\big(hg\big) = \mu(g)$. □

Lemma 4.2. $\lim_{w\to 0} \frac{\mu[0,w]}{w^{1-\gamma}} = \frac{\sigma_0}{1-\gamma}$. *Consequently, there exists $a > \frac{\sigma_0}{1-\gamma} > a' > 0$ such that*

$$a'w^{1-\gamma} \le \mu[0,w] \le aw^{1-\gamma}.$$

Proof. Use the fact that $\mu[0,w] = \int_0^w h(x)dx$ and then use Theorem A.iii). □

Lemma 4.3. *Let $w \in I_0$. Then*

$$\int_0^w \prod_{i=1}^n \eta(x_i)d\mu(x) = \mu[0,w_n].$$

Proof. Note that $f^n : [0,w_n] \to [0,w]$ is a one to one map. We have $\tilde{\mathcal{L}}^n\chi_{[0,w_n]}(x) = \prod_{i=1}^n \eta(x_i)$ as $x \in [0,w]$ and $\tilde{\mathcal{L}}^n\chi_{[0,w_n]}(x) = 0$ as $x \in [w,1]$. So by Lemma 4.1.ii),

$$\int_0^w \prod_{i=1}^n \eta(x_i)d\mu(x) = \mu\big(\tilde{\mathcal{L}}^n\chi_{[0,w_n]}\big) = \mu[0,w_n]. \qquad \square$$

Take $\psi(x)$ such that $\eta(x) = 1 - \psi(x)$. Recall the definition (1.4) of σ_g.

Lemma 4.4. *η and ψ have the following properties:*

i) *$\psi(x) = \frac{1}{h(fx)}\sigma_h(x)$ if $x \in I_0$;*

ii) *$\lim_{x\to 0} \frac{\psi(x)}{x^\gamma} = 1$;*

iii) *$\lim_{x\to 0} \eta(x) = 1$, and therefore η is continuous on each I_q;*

iv) *$0 \le \eta(x) \le 1$, and $\eta(x) = 1$ if and only if $x = 0$;*

v) *$\psi(x)$ is strictly increasing and $\eta(x)$ is strictly decreasing on P_0;*

vi) *$\forall x \in P_0$ and $\bar{x} \in I\backslash P_0$, $\eta(x) > \eta(\bar{x})$, if P_0 is small enough.*

Proof. Since $\mathcal{L}h = h$, $h(fx) = \frac{h(x)}{f'(x)} + \sigma_h(x)$ for $x \in P_0$. So $\eta(x) = \frac{h(x)}{f'(x)h(fx)} = 1 - \frac{1}{h(fx)}\sigma_h(x)$. This implies i).

Part ii) follows from Part i) and Theorem A.iii).

By Part ii), $\eta = 1 - x^\gamma + o(x^\gamma)$. So η is continuous at 0. By the definition, it is continuous at all other points.

By Lemma 4.1.i), $\sum\limits_{\hat{x}_1 \in f^{-1}x} \eta(\hat{x}_1) = 1$, so $0 \le \eta(\hat{x}_1) \le 1$. Then Part iv) is clear.

To get Part v), we use Part i) and then compare Lemma 3.5 and Corollary 2.4, from which we see that $h(x)$ changes in a faster rate than $\sigma_h(x)$.

Part vi) simply follows from Part iv) and v). $\square$

By part i) of the lemma, for $x \in I_0$ we can write $\tilde{\mathcal{L}}g(x)$ as

$$\tilde{\mathcal{L}}g(x) = \eta(x_1)g(x_1) + \psi(x_1)\bar{g}(x_1) = \big(1 - \psi(x_1)\big)g(x_1) + \psi(x_1)\bar{g}(x_1), \tag{4.1}$$

where $\bar{g}(x_1)$ is the average of $\{g(\bar{x}_1)\}$ with weight $\{\eta(\bar{x}_1)\}$, $\bar{x}_1 \in f^{-1}x \backslash I_0$, i.e.

$$\bar{g}(x_1) = \frac{\sum\limits_{\bar{x}_1 \in f^{-1}x\backslash I_0} \eta(\bar{x}_1)g(\bar{x}_1)}{\sum\limits_{\bar{x}_1 \in f^{-1}x\backslash I_0} \eta(\bar{x}_1)} = \frac{\sum\limits_{\bar{x}_1 \in f^{-1}x\backslash I_0} \frac{h(\bar{x}_1)}{f'(\bar{x}_1)} g(\bar{x}_1)}{\sum\limits_{\bar{x}_1 \in f^{-1}x\backslash I_0} \frac{h(\bar{x}_1)}{f'(\bar{x}_1)}}. \tag{4.2}$$

The second part of the lemma says that if higher order terms are ignored, then $\psi(x_1) \approx x_1^\gamma$ and therefore $\tilde{\mathcal{L}}$ has the form

$$\tilde{\mathcal{L}}g(x) \approx \big(1 - x_1^\gamma\big)g(x_1) + x_1^\gamma \bar{g}(x_1). \tag{4.3}$$

Lemma 4.5. *Let $g_n(x) = \tilde{\mathcal{L}}^n g(x)$. Then for any $x \in P_0$,*

$$g_n(x) = g(x_n)\prod_{i=1}^{n} \eta(x_i) + g_n^*(x),$$

where

$$g_n^*(x) = \sum_{j=1}^{n} \bar{g}_{n-j}(x_j)\psi(x_j)\prod_{i=1}^{j-1} \eta(x_i).$$

Proof. Use induction. By (4.1) the result is true for $n = 1$. Suppose it is true for some n. Then

$$\tilde{\mathcal{L}}g_n(x) = \eta(x_1)g(x_{n+1})\prod_{i=2}^{n+1} \eta(x_i) + \eta(x_1)g_n^*(x_1) + \psi(x_1)\bar{g}_n(x_1).$$

Since

$$\begin{aligned}\eta(x_1)g_n^*(x_1) &= \eta(x_1)\sum_{j=1}^{n} \bar{g}_{n-j}(x_{j+1})\psi(x_{j+1})\prod_{i=2}^{j} \eta(x_i)\\ &=\eta(x_1)\sum_{j=2}^{n+1} \bar{g}_{n+1-j}(x_j)\psi(x_j)\prod_{i=2}^{j-1} \eta(x_i) = \sum_{j=2}^{n+1} \bar{g}_{n+1-j}(x_j)\psi(x_j)\prod_{i=1}^{j-1} \eta(x_i),\end{aligned}$$

we get

$$\eta(x_1)g_n^*(x_1) + \psi(x_1)\bar{g}_n(x_1) = \sum_{j=1}^{n+1} \bar{g}_{n+1-j}(x_j)\psi(x_j)\prod_{i=1}^{j-1}\eta(x_i),$$

which is equal to $g_{n+1}^*(x)$. This completes the proof. $\square$

Lemma 4.6. *Let $x, y \in P_0$ with $x > y$. If $\bar{g}_i(x) \geq 0 \ \ \forall 0 \leq i \leq n-1$, then*

$$\begin{aligned}\tilde{\mathcal{L}}^n g(x) - \tilde{\mathcal{L}}^n g(y) \geq & g(x_n)\prod_{i=1}^{n}\eta(x_i) - g(y_n)\prod_{i=1}^{n}\eta(y_i) \\ & + \sum_{j=1}^{n}\Big(\bar{g}_{n-j}(x_j) - \bar{g}_{n-j}(y_j)\Big)\psi(x_j)\prod_{i=1}^{j-1}\eta(x_i).\end{aligned}$$

Proof. By Lemma 4.5, we only need prove

$$g_n^*(x) - g_n^*(y) \geq \sum_{j=1}^{n}\Big(\bar{g}_{n-j}(x_j) - \bar{g}_{n-j}(y_j)\Big)\psi(x_j)\prod_{i=1}^{j-1}\eta(x_i).$$

Note that

$$\begin{aligned}g_n^*(x) - g_n^*(y) = & \sum_{j=1}^{n}\Big(\bar{g}_{n-j}(x_j) - \bar{g}_{n-j}(y_j)\Big)\psi(x_j)\prod_{i=1}^{j-1}\eta(x_i) \\ & + \sum_{j=1}^{n}\bar{g}_{n-j}(y_j)\Big(\psi(x_j)\prod_{i=1}^{j-1}\eta(x_i) - \psi(y_j)\prod_{i=1}^{j-1}\eta(y_i)\Big).\end{aligned}$$

We only need prove that

$$\psi(x_j)\prod_{i=1}^{j-1}\eta(x_i) = \frac{\sigma_h(x_j)}{h(x_{j-1})}\cdot\frac{h(x_{j-1})}{(f^{j-1})'(x_{j-1})h(x)} = \frac{\sigma_h(x_j)}{(f^{j-1})'(x_{j-1})h(x)}$$

is increasing, where the first inequality follows from Lemma 4.4.i) and the definition of $\eta(x)$. By Proposition 2.3 and Corollary 2.4, both $(f^{j-1})'(y_{j-1})/(f^{j-1})'(x_{j-1})$ and $\sigma_h(x_j)/\sigma_h(x_j)$ are bounded by $1 + Jd(x, y)$. Hence by Lemma 3.5, we see that $h(x)$ decreasing faster than $\sigma_h(x_j)$ and $(f^{j-1})'(x_{j-1})$ if P_0 is small enough. $\square$

Proposition 4.7. *Given $\beta_- < \beta < \beta_+$, we can choose P_0 sufficiently small such that for any $x \in P_0$,*

i) *if $x = x_0 \leq \left(\frac{\beta}{r}\right)^{\beta}$ for some $r > 0$, then $\prod_{i=1}^{n}\eta(x_i) \geq \left(\frac{r}{r+n}\right)^{\beta_+}$;*

ii) *if $x = x_0 \geq \left(\frac{\beta}{r}\right)^{\beta}$ for some $r > 0$, then $\prod_{i=1}^{n}\eta(x_i) \leq \left(\frac{r}{r+n}\right)^{\beta_-}$.*

Proof. Take $\beta_+ > \beta'_+ > \beta''_+ > \beta$. Let P_0 be small enough such that for any $x = \left(\frac{\beta'_+}{r}\right)^\beta \in P_0$, $1 - \psi(x) \ge 1 - \frac{\beta'_+}{\beta''_+} x^\gamma$ and $1 - \left(\frac{\beta'_+}{r}\right) \ge \left(1 - \frac{1}{r}\right)^{\beta_+}$. Hence, using Lemma 2.2 for β''_+, we have

$$1 - \psi(x_i) \ge 1 - \frac{\beta'_+}{\beta''_+} x_i^\gamma \ge 1 - \frac{\beta'_+}{r+i} \ge \left(1 - \frac{1}{r+i}\right)^{\beta_+} = \left(\frac{r+i-1}{r+i}\right)^{\beta_+}.$$

Taking product we get the result of Part i).

Part ii) can be proved in a similar way. □

We denote

$$\tilde{\Delta}(x,y) = \begin{cases} 1 + \frac{\tilde{J}_0}{x} d(x,y), & \forall x \in P_0,\ y \in B\big(x, \rho(x)\big); \\ 1 + \tilde{J} d(x,y), & \forall x \in I \backslash P_0,\ y \in B\big(x, \rho(x)\big), \end{cases}$$

where $\tilde{J}, \tilde{J}_0 > 0$ are constants to be determined by the following lemma.

Lemma 4.8. *There exist constants $\tilde{J}, \tilde{J}_0 > 0$ such that $\forall x \in I$, $y \in B(x, \rho(x))$,*

i) *if $x_1 \in f^{-1}x$, $y_1 \in f^{-1}y \cap B(x_1, \rho(x_1))$, then*

$$\tilde{\Delta}(x_1, y_1) \cdot \frac{\eta(y_1)}{\eta(x_1)} \le \tilde{\Delta}(x,y);$$

ii) *if $x_n \in f^{-n}x$, $y_n \in f^{-n}y \cap B_n(x_n, \rho)$, then*

$$\prod_{i=1}^{n} \frac{\eta(f^i y_n)}{\eta(f^i x_n)} \le \tilde{\Delta}(x,y) \qquad \forall n > 0;$$

iii) *if a function g satisfies $g(\hat{y}_1) \le g(\hat{x}_1)\tilde{\Delta}(\hat{x}_1, \hat{y}_1)$ $\forall \hat{x}_1 \in f^{-1}x$, $\hat{y}_1 \in f^{-1}y \cap B(\hat{x}_1, \rho(\hat{x}_1))$, then $\tilde{\mathcal{L}}g(y) \le \tilde{\mathcal{L}}g(x) \cdot \tilde{\Delta}(x,y)$.*

Proof. Notice that $\frac{\eta(y)}{\eta(x)} = \frac{h(y)}{h(x)} \cdot \frac{f'(x)}{f'(y)} \cdot \frac{h(fx)}{h(fy)}$. So if we take $\tilde{J}, \tilde{J}_0 > 0$ such that

$$\tilde{\Delta}(x,y) \ge \Delta(x,y)^2 \Delta(fx, fy),$$

then the rest is the same as in the proof of Proposition 2.3 and Lemma 3.3. □

Remark. Recall the remark after Proposition 2.3. We also have that if $f^{n-1}x_n \in I \backslash I_0$, then

$$\prod_{i=1}^{n} \frac{\eta(f^i y_n)}{\eta(f^i x_n)} \le 1 + \tilde{J} d(x,y)$$

for some $\tilde{J} > 0$ even if $x \in P_0$.

Recall the definition of $\bar{g}(x)$ in (4.2).

Lemma 4.9. *There exists a constant $\bar{J} > 0$ such that for all $x, y \in I_0$, with $d(x,y) \le \bar{\rho}$, if $g(\bar{y}) \le g(\bar{x})\Delta(\bar{x},\bar{y})$ $\forall \bar{x} \in f^{-1}(fx)\backslash I_0$, $\bar{y} \in f^{-1}(fy) \cap B(\bar{x}, \rho(\bar{x}))$, then*

$$\bar{g}(y) \le \bar{g}(x)\big(1 + \bar{J}d(x,y)\big).$$

Proof. Clearly, $\eta(\bar{y})g(\bar{y}) \le \eta(\bar{x})g(\bar{x})\tilde{\Delta}(\bar{x},\bar{y})^2$. Hence, by (4.2),

$$\bar{g}(y) = \frac{\sum\limits_{\bar{y}\in f^{-1}(fy)\backslash I_0} \eta(\bar{y})g(\bar{y})}{\sum\limits_{\bar{y}\in f^{-1}(fy)\backslash I_0} \eta(\bar{y})} \le \frac{\sum\limits_{\bar{x}\in f^{-1}(fx)\backslash I_0} \eta(\bar{x})g(\bar{x})\tilde{\Delta}(x,y)^2}{\sum\limits_{\bar{x}\in f^{-1}(fx)\backslash I_0} \eta(\bar{x})\tilde{\Delta}(\bar{x},\bar{y})^{-1}} = \bar{g}(x)\max\{\tilde{\Delta}(\bar{x},\bar{y})^3\},$$

where max is taken over all pairs $\bar{x} \in f^{-1}(fx)\backslash I_0$ and $\bar{y} \in f^{-1}(fy) \cap B(\bar{x}, \rho(\bar{x}))$. So the result follows by choosing $\bar{J} > 0$ such that $1 + \bar{J}d(x,y) \ge \big(1 + \tilde{J}d(\bar{x},\bar{y})\big)^3$. $\square$

5. Convergent Rate

The main result in this section is Proposition 5.2, which shows that the rate of convergence $\tilde{\mathcal{L}}^n g \to \mu(g)$ is polynomial. This proposition plays a key role for the proof of Theorem B. Since the proof is long, we put some lemmas in next section.

From now on we denote $g_n(x) = \tilde{\mathcal{L}}^n g(x)$.

For any $b_+ \in (0,1)$, define a function $\Gamma(x,y) = \Gamma_{b_+}(x,y)$ by

$$\Gamma(x,y) = \begin{cases} 1 + \frac{K_0}{x} d(x,y) & \forall x \in P_0,\ y \in B\big(x, \rho(x)\big); \\ 1 + Kd(x,y), & \forall x \in I\backslash P_0,\ y \in B\big(x, \rho(x)\big), \end{cases}$$

where $K, K_0 > 0$ are constants chosen as in the following lemma.

Lemma 5.1. *There exist constants $K, K_0 > 0$ such that for any $x \in I$, $y \in B(x, \rho(x))$,*

i) *if $g(x) \le b_+$, $g(x) \le g(y)\tilde{\Delta}(y,x)$ and $g(y) \le g(x)\tilde{\Delta}(x,y)$, then*

$$1 - g(y) \le \big(1 - g(x)\big)\Gamma(x,y);$$

ii) *if $1 - g(\bar{y}_1) \le \big(1 - g(\bar{x}_1)\big)\Gamma(\bar{x}_1,\bar{y}_1)$ $\forall \bar{x}_1 \in f^{-1}x$, $\bar{y}_1 \in f^{-1}y \cap B(\bar{x}_1, \rho(\bar{x}_1))$, then*

$$1 - \tilde{\mathcal{L}}g(y) \le \big(1 - \tilde{\mathcal{L}}g(x)\big)\Gamma(x,y);$$

iii) *there exist constant $\bar{K} > 0$ such that for all $x, y \in I_0$ with $d(x,y) \le \bar{\rho}$, if $1 - g(\bar{y}) \le \big(1 - g(\bar{x})\big)\Gamma(\bar{x},\bar{y})$ $\forall \bar{x} \in f^{-1}(fx)\backslash I_0$, $\bar{y} \in f^{-1}(fy) \cap B(\bar{x}, \rho(\bar{x}))$, then*

$$1 - \bar{g}(y) \le \big(1 - \bar{g}(x)\big)\big(1 + \bar{K}d(x,y)\big).$$

Proof. Since $g(x) \le g(y)\tilde\Delta(y,x) = g(y) + g(y)\big(\tilde\Delta(y,x)-1\big)$, we have

$$1-g(y) \le 1-g(x)+g(y)\big(\tilde\Delta(y,x)-1\big) \le \big(1-g(x)\big)\Big(1+\frac{g(y)}{1-b_+}\big(\tilde\Delta(y,x)-1\big)\Big)$$

$$\le \big(1-g(x)\big)\big(1+\tilde\Delta(y,x)-1\big)^{\max\{1,\frac{g(y)}{1-b_+}\}} \le \big(1-g(x)\big)\big(\tilde\Delta(y,x)\big)^{\max\{1,\frac{g(y)}{1-b_+}\}}.$$

Note that for $x \in P_0$, $d(x,y) \le \rho(x) = O(x^{1+\gamma})$, and $g(y) \le g(x)\tilde\Delta(x,y) \le b_+\tilde\Delta(x,y)$. So $g(y)$ is bounded. Hence, it is clear that K_0 and K exist. This is Part i). Part ii) and iii) follow from the same arguments as in the proof of Lemma 4.8 and Lemma 4.9. $\Box$

Proposition 5.2. *For any $0 < b_- \le b_+ < 1$, we can find arbitrarily small $v \in P_0$ such that for any continuous functions $g_+ \ge g_- > 0$ of the form*

$$g_\pm(x) = \begin{cases} A_\pm \prod_{i=0}^{k-1}\eta(x_i), & x \in [0,v]; \\ b_\pm, & x \in [v,1], \end{cases} \tag{5.1}$$

where $A_+ \ge A_- > 1$ and $k > 0$ are constants that make $\mu(g_+) \ge 1$ and $\mu(g_-) \le 1$, if a function g satisfies

(a) *$g(x) \le g_+(x)$ $\forall x \in I$, and $g(x) \ge g_-(x)$ $\forall x \le v$,*
(b) *$\mu(g) = 1$,*
(c) *$g(y) \le g(x)\tilde\Delta(x,y)$ $\forall x \in I, y \in B(x,\rho(x))$, and*
(d) *g is decreasing on $[0,v]$,*

then for all $n \ge 0$,

i) $1-g_n(x) \ge \dfrac{D'A_-}{(n+k)^{\beta-1}}$ $\forall x \in I\backslash I_0$,

ii) $1-g_n(x) \le \dfrac{DA_+}{n^{\beta-1}}$ $\forall x \in I$,

iii) $\dfrac{\bar D'A_-}{(n+k)^{\beta-1}} \le \displaystyle\int |g_n(x)-1|d\mu(x) \le \dfrac{\bar DA_+}{n^{\beta-1}}$,

where $D, D', \bar D, \bar D' > 0$ are constants only depending on f.

Proof. We divide the proof into three steps.

Step I. We choose v and construct functions $g_\pm(x)$.

Take $0 < b_- \le b_+ < 1$.

Take $u \in I_0$ with $u \le \bar\rho$, where $\bar\rho = \inf\{\rho(x) : x \in I\backslash I_0\}$, such that for all $x > u$, $\eta(x) \le \eta(u)$, and for all $x \in [u, fu]$, $y \in B(x,\rho(x))$, $\Gamma(x,y) \ge 1 + 3\bar K d(\bar x,\bar y)$ $\forall \bar x,\bar y \in I\backslash I_0$ with $d(\bar x,\bar y) \le \rho(\bar x)$. This is possible because of the definition of $\Gamma(x,y)$.

Take $v = u_m \in P_0$ for some $m > 0$, and write $v = \Big(\dfrac{\beta}{s}\Big)^\beta$. We assume first that $s \ge m$, otherwise we can choose a smaller u. Then we assume that m is large enough such that

$$\prod_{i=1}^{m}\eta(x_i) \le \frac{1}{2} \qquad \forall x \in I_0\backslash[0,u]. \tag{5.2}$$

Since $\prod_{i=1}^{n}\eta(x_i)+\sum_{j=1}^{n}\psi(x_j)\prod_{i=1}^{j-1}\eta(x_i)=1$, it implies that for any $n\ge m$,

$$\sum_{j=1}^{n}\psi(x_j)\prod_{i=1}^{j-1}\eta(x_i)\ge\frac{1}{2}\qquad\forall x\in I_0\backslash[0,u]. \tag{5.3}$$

Lastly we assume that s is large enough such that

$$c's^{\frac{\beta-1}{\beta_+-\beta+1}}\ge\max\Big\{\Big(\frac{2^{\beta_+}}{b_-}\Big)^{\frac{1}{\beta_-}},\ \Big(\frac{2a\beta^{\beta-1}}{a'b_-\beta_-^{\beta-1}}\Big)^{\frac{1}{\beta_--\beta+1}}\ \Big\}, \tag{5.4}$$

and

$$s^{\beta_--\beta+1}\ge\frac{4C_1C_2b_+}{C_3'b_-}, \tag{5.5}$$

where c' is given in (5.13), a and a' are given in Lemma 4.2, C_2 and C_3' are as in Lemma 6.1 and 6.2 respectively, and $C_1\ge\big(1-b_+\big)^{-1}$ and satisfies that if

$$\begin{aligned}1-g(y)\le 2\big(1-g(x)\big)\big(1+\bar{K}d(x,y)\big)\qquad&\forall x\in[u,fu],\ 0<y\le x;\\ 1-g(y)\le\big(1-g(x)\big)\Gamma(x,y)\qquad&\forall x\ge u,\ y\in B(x,\rho(x)),\end{aligned}$$

then

$$\max\{1-g(x),\ x\in I\}\le C_1\min\{1-g(x),\ x\ge u\}. \tag{5.6}$$

Now we choose $A_+\ge A_-\ge 1$ and $k>0$ such that

$$A_\pm\prod_{i=0}^{k-1}\eta(v_i)=b_\pm, \tag{5.7}$$

$$A_+\mu[0,v_k]+b_+\mu[v,1]\ge 1\qquad\text{and}\qquad A_-\mu[0,v_k]+b_-\mu[v,1]\le 1. \tag{5.8}$$

This is possible. In fact, by Lemma 4.7 and 4.2 we have

$$\Big(\frac{s}{s+k}\Big)^{\beta_+}\le\prod_{i=1}^{k}\eta(v_i)\le\Big(\frac{s}{s+k}\Big)^{\beta_-}, \tag{5.9}$$

$$a'\Big(\frac{\beta_-}{s+k}\Big)^{\beta-1}\le\mu[0,v_k]\le a\Big(\frac{\beta_+}{s+k}\Big)^{\beta-1}. \tag{5.10}$$

These imply $\lim_{k\to 0}\frac{\prod_{i=0}^{k-1}\eta(v_i)}{\mu[0,v_k]}=0$. So we can take k such that

$$\frac{1}{b_+}-\mu[v,1]\le\frac{\mu[0,v_k]}{\prod_{i=0}^{k-1}\eta(v_i)}\le\frac{1}{b_-}-\mu[v,1] \tag{5.11}$$

and then take $A_\pm$ such that (5.7) is satisfied.

Now we define $g_\pm(x)$ by using (5.1). Lemma 4.3 and (5.8) give

$$\mu(g_+) = A_+\mu[0, v_k] + b_+\mu[v, 1] \ge 1, \quad \mu(g_-) = A_-\mu[0, v_k] + b_-\mu[v, 1] \le 1.$$

Note that by (5.9)$-$(5.11) we can obtain

$$\frac{s^{\beta_+}}{a\beta_+^{\beta-1}}\Big(\frac{1}{b_+} - \mu[v, 1]\Big) \le (s + k)^{\beta_+-\beta+1}, \quad (s + k)^{\beta_--\beta+1} \le \frac{s^{\beta_-}}{a'\beta_-^{\beta-1}}\Big(\frac{1}{b_-} - \mu[v, 1]\Big),$$

and therefore,

$$c' s^{\frac{\beta_+}{\beta_+-\beta+1}} \le k + s, \qquad k \le cs^{\frac{\beta_-}{\beta_--\beta+1}}, \tag{5.12}$$

where $c > c' > 0$ are constants satisfying

$${c'}^{\beta_+-\beta+1} = \frac{1}{a\beta_+^{\beta-1}}\Big(\frac{1}{b_+} - \mu[v, 1]\Big), \quad c^{\beta_--\beta+1} = \frac{1}{a'\beta_-^{\beta-1}}\Big(\frac{1}{b_-} - \mu[v, 1]\Big). \tag{5.13}$$

Hence by (5.4) we have

$$k + s \ge c' s^{\frac{\beta-1}{\beta_+-\beta+1}} \cdot s \ge \Big(\frac{2^{\beta_+}}{b_-}\Big)^{\frac{1}{\beta_-}} \cdot s \ge 2s, \quad \text{i.e.} \quad k \ge s. \tag{5.14}$$

Moreover, by (5.7) and (5.9),

$$b_\pm\Big(\frac{s + k}{s}\Big)^{\beta_-} \le A_\pm \le b_\pm\Big(\frac{s + k}{s}\Big)^{\beta_+}, \tag{5.15}$$

and therefore by (5.14),

$$A_- \ge b_-\Big(\frac{s + k}{s}\Big)^{\beta_-} \ge 2^{\beta_+} > 1. \tag{5.16}$$

Step II. We prove that any function g satisfying condition (a)-(d) has the following property:

($\mathcal{A}_n$) $1 - g_n(x) > 0$ $\forall x \ge u$;
($\mathcal{B}_n$) $\max\{1 - g_n(x),\ x \in I\} \le C_1 \min\{1 - g_n(x),\ x \ge u\}$.

First we consider the case $0 \le n \le m$. Since $1-g(x) \ge 1-b_+ > 0$ $\forall x \ge v = u_m$, by Lemma 4.1.i), $1 - g_n(x) \ge 1 - b_+ > 0$ $\forall x \ge u \ge f^n v$. We get ($\mathcal{A}_n$). Since $C_1 \ge \big(1 - b_+\big)^{-1}$, ($\mathcal{B}_n$) follows.

Now we consider the cases $n > m$. We only need prove the following:

($\mathcal{A}_n^*$) $1 - g_n(x) > 2g(x_n)\prod_{i=1}^n \eta(x_i)$ $\forall x \in [u, fu]$;
($\mathcal{B}_n'$) $1 - g_n(y) \le 2\big(1 - g_n(x)\big)\big(1 + \bar{K}d(x, y)\big)$ $\forall x \in [u, fu]$, $0 < y \le x$;
($\mathcal{B}_n''$) $1 - g_n(y) \le \big(1 - g_n(x)\big)\Gamma(x, y)$ $\forall x \ge u,\ y \in B(x, \rho(x))$.

In fact, by the definition of $\tilde{\mathcal{L}}$, we know that $(\mathcal{A}_{n-1})$ and $(\mathcal{A}_n^*)$ imply $(\mathcal{A}_n)$. Also, by $(\mathcal{B}'_n)$, $(\mathcal{B}''_n)$ and (5.6), we can get $(\mathcal{B}_n)$.

To prove $(\mathcal{A}_n^*)$, $(\mathcal{B}'_n)$ and $(\mathcal{B}''_n)$, we use induction. Assume $(\mathcal{B}_j)$ are true for all $0 \le j \le n-1$. Then by Lemma 6.3 and the choice of C_1, $(\mathcal{A}_n^*)$ is true.

Note that $(\mathcal{B}''_j)$ holds for any $j = 0, 1, \cdots, m$ because of Lemma 5.1.i), ii) and the fact that $1 - g(x) \ge 1 - b_+ > 0$ $\forall x \ge v$. So we may assume $(\mathcal{A}_j)$ and $(\mathcal{B}''_j)$ for all $j = 0, 1, \cdots, n-1$. Hence, if we assume $(\mathcal{A}_n^*)$ in addition, then by Lemma 6.4 and Lemma 6.5, $(\mathcal{B}'_n)$ and $(\mathcal{B}''_n)$ hold respectively.

Step III. We prove that g_n satisfies i)-iii).

Since $\mu(g_n) = \mu(g) = 1$,

$$\int_{\{g_n>1\}} \big(g_n(x)-1\big)d\mu(x) = \int_{\{g_n<1\}} \big(1-g_n(x)\big)d\mu(x) = \frac{1}{2}\int |g_n(x)-1|d\mu(x). \quad (5.17)$$

Then the first inequality of Part iii) follows immediately from Lemma 6.2 with $\bar{D}' = 2C'_3$. By using $(\mathcal{A}_n)$ $\forall n > 0$, we get that the upper bound estimate in Part iii) follows from Lemma 6.6 with $\bar{D} = 2C_4$.

If we use $(\mathcal{B}_n)$, then

$$\begin{aligned} &\int_{\{g_n<1\}} \big(1 - g_n(x)\big)d\mu(x) \le \max\{1 - g(x) :\ x \in I\} \\ \le &C_1 \min\{1 - g(x) :\ x \ge u\} \le \frac{C_1}{\mu(I\backslash I_0)} \int_{\{g_n<1\}} \big(1 - g_n(x)\big)d\mu(x). \end{aligned} \quad (5.18)$$

Considering (5.17) and the results in Part iii), we get i) with $D' = \big(2C_1\big)^{-1}\bar{D}' = C_1^{-1}C'_3$, and get ii) with $D = \big(2\mu(I\backslash I_0)\big)^{-1}C_1\bar{D} = \big(\mu(I\backslash I_0)\big)^{-1}C_1C_4$. $\square$

6. Some Supplementary Lemmas

In this section we prove lemmas which are used for the proof of Proposition 5.2.

Lemma 6.1. *There exists $C_2 > 0$ such that for any $x > u$,*

$$\prod_{i=1}^{k+n} \eta(x_i) \le C_2 \frac{1}{k^{\beta_- - \beta + 1}} \cdot \frac{1}{(n+k)^{\beta-1}} \qquad \forall n, k > 0.$$

Proof. By Lemma 4.7, for $x^* = \Big(\frac{\beta_-}{r^*}\Big)^{\beta} \in P_0$ fixed, $\prod_{i=1}^{k+n} \eta(x_i^*) \le \Big(\frac{r^*}{r^* + k + n}\Big)^{\beta_-}$. So the result is clear for this x^*. Since by Lemma 4.4.iv) $\eta(x)$ is smaller outside P_0 than inside P_0, the result holds for all $x \in I_0\backslash P_0$ as well. $\square$

Lemma 6.2. *Let $C_3' = 2^{-\beta} a' \beta_-^{\beta-1}$, where a' is given in Lemma 4.2. Then*

$$\int_{\{g_n>1\}} \big(g_n(x)-1\big)d\mu(x) \ge C_3' A_- \frac{1}{(n+k)^{\beta-1}} \qquad \forall n>0.$$

Proof. Take $t>0$ such that

$$\Big(\frac{t}{t+k}\Big)^{\beta_+} = \frac{1}{b_-}\Big(\frac{s}{s+k}\Big)^{\beta_-}. \tag{6.1}$$

Clearly, $s \le t$. Also, by (5.16) the right side is no more than $\dfrac{1}{2^{\beta_+}}$. So $\dfrac{t}{t+k} \le \dfrac{1}{2}$. We get $t \le k$.

Take

$$z^{(n)} = \Big(\frac{\beta}{t(1+\frac{n}{k})}\Big)^{\beta}. \tag{6.2}$$

We claim

$$[0, z^{(n)}] \subset \{x : g_n(x) \ge 1\} \qquad \forall n \ge 0. \tag{6.3}$$

In fact, for any $x \le z^{(n)}$, by Proposition 4.7 and (5.9),

$$\prod_{i=1}^{k+n} \eta(x_i) \ge \Big(\frac{t(1+\frac{n}{k})}{t(1+\frac{n}{k})+k+n}\Big)^{\beta_+} = \Big(\frac{t}{t+k}\Big)^{\beta_+} = \frac{1}{b_-}\Big(\frac{s}{s+k}\Big)^{\beta_-} \ge \frac{1}{b_-}\prod_{i=1}^{k}\eta(v_i).$$

Then by (5.1), (4.1) and (5.7), $g_n(x) \ge A_- \displaystyle\prod_{i=1}^{k+n}\eta(x_i) \ge \frac{A_-}{b_-}\prod_{i=1}^{k}\eta(v_i) = 1$.

Now using Lemma 4.3 we have

$$\begin{aligned}\int_{\{g_n>1\}} \big(g_n(x)-1\big)d\mu(x) \ge& A_- \int_0^{z^{(n)}} \prod_{i=1}^{k+n}\eta(x_i)d\mu(x) - \mu[0,z^{(n)}] \\ =& A_-\mu[0, z^{(n)}_{n+k}] - \mu[0, z^{(n)}].\end{aligned}$$

Since $k>t$, by (6.2) and Lemma 2.2, $z^{(n)}_{n+k} \ge \Big(\dfrac{\beta_-}{t(1+\frac{n}{k})+k+n}\Big)^{\beta} \ge \Big(\dfrac{\beta_-}{2(k+n)}\Big)^{\beta}$.

Then by Lemma 4.2,

$$A_-\mu[0, z^{(n)}_{n+k}] \ge A_- a'\big(z^{(n)}_{n+k}\big)^{\beta-1} \ge \frac{A_- a' \beta_-^{\beta-1}}{2^{\beta-1}}\Big(\frac{1}{k+n}\Big)^{\beta-1}.$$

So the result follows if we show $A_-\mu[0, z^{(n)}_{n+k}] \ge 2\mu[0, z^{(n)}]$.

Note that $\dfrac{z^{(n)}_{n+k}}{z^{(n)}} \ge \Big(\dfrac{\beta_- t}{\beta(t+k)}\Big)^{\beta}$. Using (5.16) and the fact $t>s$, we can get

$$A_-\frac{\mu[0, z^{(n)}_{n+k}]}{\mu[0, z^{(n)}]} \ge b_-\Big(\frac{s+k}{s}\Big)^{\beta_-} \cdot \frac{a'}{a}\Big(\frac{\beta_- t}{\beta(t+k)}\Big)^{\beta-1} \ge \frac{a' b_-}{a}\Big(\frac{\beta_-}{\beta}\Big)^{\beta-1}\Big(\frac{s+k}{s}\Big)^{\beta_- -\beta+1}.$$

The right side is greater than or equal to 2 because by (5.12) and (5.4),

$$\Big(\frac{k+s}{s}\Big)^{\beta_- -\beta+1} \ge \Big(c' s^{\frac{\beta-1}{\beta_+ -\beta+1}}\Big)^{\beta_- -\beta+1} \ge \frac{2a}{a' b_-}\Big(\frac{\beta}{\beta_-}\Big)^{\beta-1}. \qquad \square$$

Lemma 6.3. *Let $n > m$. Suppose for all $0 \le j \le n-1$,*

$$\max\{1-g_j(x),\ x \in I\} \le C_1 \min\{1-g_j(x),\ x \ge u\}.$$

Then $1-g_n(x) > 2g(x_n)\prod_{i=1}^{n}\eta(x_i)$ $\forall x \in [u, fu]$.

Proof. By (5.17), (5.18) and Lemma 6.2, we have that for all $1 \le j \le n$,

$$\begin{aligned} 1-g_j(x) &\ge \frac{1}{C_1}\int_{\{g_j>1\}}\big(g_j(x)-1\big)d\mu(x) \\ &\ge \frac{C_3'A_-}{C_1}\frac{1}{(k+j)^{\beta-1}} \ge \frac{C_3'A_-}{C_1}\frac{1}{(k+n)^{\beta-1}} \qquad \forall x \ge u. \end{aligned}$$

So the same inequality is true for $1-\bar g_j(x)$. By Lemma 4.5 and (5.3),

$$1-g_n(x) > \sum_{j=1}^{n}\big(1-\bar g_{n-j}(x_j)\big)\psi(x_j)\prod_{i=1}^{j-1}\eta(x_i) \ge \frac{C_3'A_-}{2C_1}\frac{1}{(k+n)^{\beta-1}}.$$

On the other hand, since $n > m$, $g(x_n) \le A_+\prod_{i=1}^{k}\eta(x_i)$. So by Lemma 6.1,

$$g(x_n)\prod_{i=1}^{n}\eta(x_i) \le A_+\prod_{i=1}^{k+n}\eta(v_i) \le A_+C_2\frac{1}{k^{\beta_--\beta+1}}\frac{1}{(k+n)^{\beta-1}}.$$

Now, considering (5.7) and (5.14) we have

$$\frac{1-g_n(x)}{2g(x_n)\prod_{i=1}^{n}\eta(x_i)} \ge \frac{C_3'A_-}{2^2C_1C_2A_+}k^{\beta_--\beta+1} \ge \frac{C_3'b_-}{4C_1C_2b_+}s^{\beta_--\beta+1}.$$

By (5.5) it is greater than or equal to 1. □

Lemma 6.4. *Let $n > m$. Suppose $g(x)$ is decreasing on $[0, v]$. Suppose further*

(i) $1-g_j(x) > 0$ $\forall 0 \le j \le n-1,\ x \ge u,$

(ii) $1-g_j(y) \le \big(1-g_j(x)\big)\Gamma(x,y)$, $\forall 0 \le j \le n-1,\ x \ge u,\ y \in B(x,\rho(x))$, *and*

(iii) $1-g_n(x) \ge 2g(x_n)\prod_{i=1}^{n}\eta(x_i)$ $\forall x \in [u, fu]$.

Then for all $x \in [u, fu]$ with $1-g_n(x) > 0$,

$$1-g_n(y) \le 2\big(1-g_n(x)\big)\big(1+\bar K d(x,y)\big) \qquad \forall 0 < y \le x.$$

Proof. By Supposition (ii) and Lemma 5.1.iii),

$$\big(1-\bar g_{n-j}(x_j)\big)-\big(1-\bar g_{n-j}(y_j)\big) \ge -\bar K d(x,y)\cdot\big(1-\bar g_{n-j}(x_j)\big). \tag{6.4}$$

Using Lemma 4.5 for the function $1-g(x)$ and then using Supposition (iii), we have

$$\begin{aligned}\sum_{j=1}^{n}\big(1-\bar g_{n-j}(x_j)\big)\psi(x_j)\prod_{i=1}^{j-1}\eta(x_i)&=1-g_n(x)-\big(1-g(x_n)\big)\prod_{i=1}^{n}\eta(x_i)\\ &<1-g_n(x)+g(x_n)\prod_{i=1}^{n}\eta(x_i)\le\frac{3}{2}\big(1-g_n(x)\big)\le 2\big(1-g_n(x)\big).\end{aligned}\tag{6.5}$$

Therefore, by using Lemma 4.6 for the function $1-g(x)$, we obtain

$$\begin{aligned}&\big(1-g_n(x)\big)-\big(1-g_n(y)\big)\\ \ge&\big(1-g(x_n)\big)\prod_{i=1}^{n}\eta(x_i)-\big(1-g(y_n)\big)\prod_{i=1}^{n}\eta(y_i)-2\bar K\big(1-g_n(x)\big)d(x,y).\end{aligned}\tag{6.6}$$

If $1-g(y_n)\le 0$, then either $1-g(x_n)\ge 0$ or $0\ge 1-g(x_n)\ge 1-g(y_n)$. Since $\eta(x)$ is decreasing, (6.6) becomes

$$\big(1-g_n(x)\big)-\big(1-g_n(y)\big)\ge -2\bar K\big(1-g_n(x)\big)d(x,y).\tag{6.7}$$

Then the result follows.

If $1-g(y_n)\ge 0$, then $0\le 1-g(y_n)\le 1-g(x_n)\le 1-g_n(x)$. Since $\eta(x_i)>0$ and $\eta(y_i)<1$, (6.6) becomes

$$\big(1-g_n(x)\big)-\big(1-g_n(y)\big)\ge -\big(1-g_n(x)\big)-2\bar K\big(1-g_n(x)\big)d(x,y).$$

This is the result of the lemma. □

Lemma 6.5. *Suppose all conditions in Lemma 6.4 are satisfied. Then*

$$1-g_n(y)\le\big(1-g_n(x)\big)\Gamma(x,y)\qquad\forall x\in[u,fu],\ y\in B(x,\rho(x)).$$

Proof. First we assume $y\le x$, The same argument as in the proof of above lemma tells that (6.5) holds. Further, if $1-g(y_n)\le 0$, then (6.7) follows as well and therefore the result is true. So we consider the case $1-g(y_n)\ge 0$. Note that $g(y_n)\ge g(x_n)\ge g_n(x)$. By Lemma 4.8.ii) and (5.2),

$$\begin{aligned}&\big(1-g(x_n)\big)\prod_{i=1}^{n}\eta(x_i)-\big(1-g(y_n)\big)\prod_{i=1}^{n}\eta(y_i)\ge\big(1-g(y_n)\big)\Big(\prod_{i=1}^{n}\eta(x_i)-\eta(y_i)\Big)\\ \ge&-\big(1-g(y_n)\big)\prod_{i=1}^{n}\eta(x_i)\big(\tilde\Delta(x,y)-1\big)\ge-\big(1-g_n(x)\big)\frac{\tilde\Delta(x,y)-1}{2}.\end{aligned}$$

So by (6.6),

$$\begin{aligned}\big(1-g_n(x)\big)-\big(1-g_n(y)\big)&\ge-\big(1-g_n(x)\big)\Big(\frac{\tilde\Delta(x,y)-1}{2}+\frac{3\bar Kd(x,y)}{2}\Big)\\ &\ge-\big(1-g_n(x)\big)\Big(\Gamma(x,y)-1\Big),\end{aligned}$$

where the last step follows from the choice of u. This is the result.

Now we assume $y \geq x$. We use Lemma 4.6 for the function $g(x)$, while interchange the roles of x and y, and replace $\bar{g}_{n-j}(x_j) - \bar{g}_{n-j}(y_j)$ by $\big(1 - \bar{g}_{n-j}(y_j)\big) - \big(1 - \bar{g}_{n-j}(y_j)\big)$, to get

$$\begin{aligned} g_n(x) - g_n(y) \leq g(x_n) \prod_{i=1}^{n} \eta(x_i) - g(y_n) \prod_{i=1}^{n} \eta(y_i) \\ + \sum_{j=1}^{n} \Big[\big(1 - \bar{g}_{n-j}(y_j)\big) - \big(1 - \bar{g}_{n-j}(x_j)\big)\Big] \psi(x_j) \prod_{i=1}^{j-1} \eta(x_i). \end{aligned} \tag{6.8}$$

Since $g(y_n) \leq g(x_n)$ and $\eta(y_n) \leq \eta(x_n)$, by Lemma 4.8.i) and Supposition (iii),

$$\begin{aligned} g(x_n) \prod_{i=1}^{n} \eta(x_i) - g(y_n) \prod_{i=1}^{n} \eta(y_i) \leq & \Big[g(y_n) \prod_{i=1}^{n} \eta(y_i)\Big] \big(\tilde{\Delta}(x,y) - 1\big) \\ \leq & \Big[g(x_n) \prod_{i=1}^{n} \eta(x_i)\Big] \big(\tilde{\Delta}(x,y) - 1\big) \leq \frac{1 - g_n(x)}{2} \big(\Gamma(x,y) - 1\big). \end{aligned}$$

Note that the arguments for (6.4) and (6.5) still hold. So (6.8) becomes

$$\begin{aligned} & \big(1 - g_n(y)\big) - \big(1 - g_n(x)\big) = g_n(x) - g_n(y) \\ \leq & (1 - g_n(x)) \Big(\frac{\Gamma(x,y) - 1}{2} + \frac{3\bar{K} d(x,y)}{2}\Big) \leq (1 - g_n(x)) \Big(\Gamma(x,y) - 1\Big). \end{aligned}$$

This completes the proof. $\square$

Lemma 6.6. *Let $C_4 = a\beta_+^{\beta-1}$. Suppose $1 - g_j(x) > 0$ $\forall 0 \leq j \leq n$, $x \geq u$. Then*

$$\int_{\{g_n > 1\}} \big(g_n(x) - 1\big) d\mu(x) \leq \frac{C_4 A_+}{n^{\beta-1}}.$$

Proof. The supposition implies that $\forall x \in I_0$,

$$\sum_{j=1}^{n} \Big[\big(\bar{g}_{n-j}(x_j) - 1\big)\Big] \psi(x_j) \prod_{i=1}^{j-1} \eta(x_i) \leq 0.$$

If $g_n(x) \geq 1 > b_+$, then $g(x_n) \geq 1 > b_+$ and therefore $x_n \leq v$. So by Lemma 4.5,

$$g_n(x) - 1 \leq \big(g(x_n) - 1\big) \prod_{i=1}^{n} \eta(x_i) < g(x_n) \prod_{i=1}^{n} \eta(x_i) \leq A_+ \prod_{i=1}^{n+k} \eta(x_i).$$

Note that $\{x : \ g_n(x) > 1\} \subset [0, u]$. Also note that $k \geq s \geq m$ and therefore $u_{n+k} = v_{n+k-m} \leq v_n$. We have

$$\begin{aligned} & \int_{\{g_n > 1\}} \big(g_n(x) - 1\big) d\mu(x) \leq A_+ \int_0^u \prod_{i=1}^{n+k} \eta(x_i) d\mu(x) = A_+ \mu[0, u_{n+k}] \\ & \leq A_+ \mu[0, v_n] \leq A_+ \cdot a v_n^{1-\gamma} \leq a A_+ \Big(\frac{\beta_+}{s+n}\Big)^{\beta-1} \leq a A_+ \Big(\frac{\beta_+}{n}\Big)^{\beta-1}. \end{aligned}$$

$\square$

7. Proofs of Theorem B and its Corollary

Proposition 7.1. *There exist $B, \bar{B} > 0$ such that for any Lipschitz function g with $\mu(g) = 1$, and for all $n > 0$,*

i) $\left|1 - \tilde{\mathcal{L}}^n g(x)\right| \le \dfrac{B}{\epsilon n^{\beta-1}}$ $\forall x \in I \backslash I_0$,

ii) $\displaystyle\int |\tilde{\mathcal{L}}^n g(x) - 1| d\mu(x) \le \frac{\bar{B}}{\epsilon n^{\beta-1}}$,

where $\epsilon > 0$ only depends on the Lipschitz constant of g.

Proof. Take $0 < b_- < b_+ < 1$, and take $v \in P_0$, $k > 0$, and functions $g_\pm$ with $\mu(g_+) > 1$ and $\mu(g_-) < 1$ as in Proposition 5.2 . Then we choose A and b such that $A \displaystyle\prod_{i=1}^{k} \eta(v_i) = b$ and such that the function $\hat{g}$ defined by

$$\hat{g}(x) = \begin{cases} A \prod_{i=0}^{k-1} \eta(x_i), & x \in [0, v]; \\ b, & x \in [v, 1] \end{cases}$$

satisfies $\mu(\hat{g}) = 1$. Then we write

$$1 - \tilde{\mathcal{L}}^n g = \frac{1}{2\epsilon}\Big[1 - \tilde{\mathcal{L}}^n\Big(\hat{g} - \epsilon\big[1 - g)\big]\Big)\Big] - \frac{1}{2\epsilon}\Big[1 - \tilde{\mathcal{L}}^n\Big(\hat{g} + \epsilon\big[1 - g\big]\Big)\Big].$$

Suppose we can find $\epsilon > 0$ such that both functions $\hat{g}(x) + \epsilon\big[1 - g(x)\big]$ and $\hat{g}(x) - \epsilon\big[1 - g(x)\big]$ satisfy the requirements (a), (c) and (d) in Proposition 5.2. By using the proposition for these functions, we can get

$$|1 - \tilde{\mathcal{L}}^n g(x)| \le \frac{1}{2\epsilon} \cdot \frac{DA_+}{n^{\beta-1}} + \frac{1}{2\epsilon} \cdot \frac{DA_+}{n^{\beta-1}} = \frac{DA_+}{\epsilon n^{\beta-1}} \qquad \forall x \in I \backslash I_0,$$

and

$$\int |\tilde{\mathcal{L}}^n g(x) - 1| d\mu(x) \le \frac{1}{2\epsilon} \cdot \frac{\bar{D}A_+}{n^{\beta-1}} + \frac{1}{2\epsilon} \cdot \frac{\bar{D}A_+}{n^{\beta-1}} = \frac{\bar{D}A_+}{\epsilon n^{\beta-1}}.$$

Therefore the result follows with $B = DA_+$ and $\bar{B} = \bar{D}A_+$.

Clearly we can find $\epsilon > 0$ such that (a) and (d) in Proposition 5.2 hold for functions $\hat{g}(x) \pm \epsilon\big(1 - g(x)\big)$. It remains to show that there exists $\epsilon > 0$ such that

$$\frac{\hat{g}(y) \pm \epsilon\big(1 - g(y)\big)}{\hat{g}(x) \pm \epsilon\big(1 - g(x)\big)} \le \tilde{\Delta}(x, y) \qquad \forall x \in I, y \in B(x, \rho(x)).$$

That is, we need

$$\frac{\tilde{\Delta}(x, y)\hat{g}(x) - \hat{g}(y)}{\big|\tilde{\Delta}(x, y)\big(1 - g(x)\big) - \big(1 - g(y)\big)\big|} \ge \epsilon > 0 \tag{7.1}$$

for all $x \in I$ and $y \in B(x, \rho(x))$.

First, we consider the case $x \in [0, v]$.

Recall the definition of $\tilde{\Delta}(x,y)$ and Lemma 4.8.i), we have

$$\Big(1 + \frac{\tilde{J}_0 d(x_k, y_k)}{x_k}\Big) \cdot \frac{\prod_{i=0}^{k-1} \eta(y_i)}{\prod_{i=0}^{k-1} \eta(x_i)} = \tilde{\Delta}(x_k, y_k) \cdot \frac{\prod_{i=0}^{k-1} \eta(y_i)}{\prod_{i=0}^{k-1} \eta(x_i)} \le \tilde{\Delta}(x,y).$$

Hence, by the definition of $\hat{g}$,

$$\begin{aligned}\tilde{\Delta}(x,y)\hat{g}(x) - \hat{g}(y) &\ge A\Big(\tilde{\Delta}(x,y) \cdot \frac{\prod_{i=0}^{k-1} \eta(x_i)}{\prod_{i=0}^{k-1} \eta(y_i)} - 1\Big) \prod_{i=0}^{k-1} \eta(y_i) \\ \ge& A\Big(1 + \frac{\tilde{J}_0 d(x_k, y_k)}{x_k} - 1\Big) \prod_{i=0}^{k-1} \eta(y_i) = A\tilde{J}_0 \frac{d(x_k, y_k)}{x_k} \prod_{i=0}^{k-1} \eta(y_i).\end{aligned}$$

Also, we have

$$\begin{aligned}&\Big|\tilde{\Delta}(x,y)\big(1 - g(x)\big) - \big(1 - g(y)\big)\Big| \\ \le&\big(\tilde{\Delta}(x,y) - 1\big)\big|1 - g(x)\big| + \big|g(y) - g(x)\big| \le \Big(\big|1 - g(x)\big| + \frac{xL_g}{\tilde{J}_0}\Big) \cdot \frac{\tilde{J}_0 d(x,y)}{x},\end{aligned}$$

where L_g is a Lipschitz constant of g.

Now we get that the left side of (7.1) is greater than or equal to

$$\frac{A \prod_{i=0}^{k-1} \eta(y_i)}{\big|1 - g(x)\big| + xL_g \tilde{J}_0^{-1}} \cdot \frac{d(x_k, y_k)}{d(x,y)} \cdot \frac{x}{x_k}.$$

It is bounded from below for all $x \in [0, v]$ and $y \in B(x, \rho(x))$ because $\big(f^k\big)'(x) \to 1$ and $\eta(y) \to 1$ as $x \to 0$.

The case $x \in [v, 1]$ can be considered similarly. $\square$

Proof of Theorem B.

First, we note that by the definition of $\tilde{\mathcal{L}}$, for any functions F and G defined on I, $\tilde{\mathcal{L}}\big((F \circ f) \cdot G\big) = F \cdot \big(\tilde{\mathcal{L}}G\big)$. Hence

$$\tilde{\mathcal{L}}^n\big((F \circ f^n) \cdot G\big) = F \cdot \big(\tilde{\mathcal{L}}^n G\big).$$

So, by using Lemma 4.1.ii) we have that

$$\begin{aligned}&\mu\Big((F \circ f^n) \cdot G\Big) - \mu(F)\mu(G) = \mu\Big(\tilde{\mathcal{L}}^n\big((F \circ f^n) \cdot G\big)\Big) - \mu\Big(F \cdot \mu(G)\Big) \\ =&\mu\Big(F \cdot \big(\tilde{\mathcal{L}}^n G\big)\Big) - \mu\Big(F \cdot \mu(G)\Big) = \mu\Big(F \cdot \big(\tilde{\mathcal{L}}^n G - \mu(G)\big)\Big).\end{aligned} \tag{7.2}$$

To prove Part i), we take Lipschitz functions F and G on $[0,1]$. Above formula gives

$$\mu\big((F \circ f^n) \cdot G\big) - \mu(F)\mu(G) \le \|F\|\mu\big(\big|\tilde{\mathcal{L}}^n G - \mu(G)\big|\big).$$

By Proposition 7.1, there exist $\bar{B} > 0$ and $\epsilon = \epsilon(G) > 0$ such that

$$\mu\big(\big|\tilde{\mathcal{L}}^n G - \mu(G)\big|\big) = \mu\big(\big|\tilde{\mathcal{L}}^n \big(G - \mu(G) + 1\big) - 1\big|\big) \le \frac{\bar{B}}{\epsilon n^{\beta-1}}.$$

So we can take $C = \bar{B}\epsilon^{-1}$.

Now we prove Part ii). Let G be any Lipschitz function satisfying the requirements (a)-(d) in Proposition 5.2 for some functions $g_-(x) \le g_+(x)$. In particular, $\mu(G) = 1$. Then we know that there exists $D' > 0$ such that for all $n > 0$,

$$1 - \tilde{\mathcal{L}}^n G(x) \ge \frac{D' A_-}{(n+k)^{\beta-1}} \qquad \forall x \in I \backslash I_0,$$

where A_- and k are described in the same proposition.

Take a Lipschitz function $F(x) \ge 0$ such that $F(x) = 0$ on I_0 and $\mu(F) > 0$. Then by (7.2) we have

$$\begin{aligned}&\big|\mu\big((F \circ f^n) \cdot G\big) - \mu(F)\mu(G)\big| = \big|\mu\big(\chi_{I\backslash I_0} \cdot F \cdot (\tilde{\mathcal{L}}^n G - 1)\big)\big| \\ \ge&\mu(F) \min_{x \in I\backslash I_0} \{1 - \tilde{\mathcal{L}}^n G(x)\} \ge \mu(F) \frac{D' A_-}{(n+k)^{\beta-1}}.\end{aligned}$$

Now the result follows with $C' = (k+1)^{-(\beta-1)} D' A_- \mu(F)$. $\square$

Recall that $E^{(j)}$ is the element of ξ_j containing 0.

Lemma 7.2. *There exist $l > 0$ such that for all $j \ge l$, if a function g satisfies*
  (a) $g(x) > 0$ *as* $x \in E^{(j)}$ *and* $g(x) = 0$ *as* $x \notin E^{(j)}$,
  (b) $\int_{E^{(j)}} g d\mu = 1$, *and*
  (c) $g(y) \le g(x)\big(1 + \tilde{J} d(x,y)\big)$ $\forall x, y \in E^{(j)}$,

then $\forall n > 0$,

  i) $1 - \tilde{\mathcal{L}}^{n+j} g(x) \ge \dfrac{D' A_-}{(n+j)^{\beta-1}}$ $\forall x \in I \backslash I_0$,
  ii) $1 - \tilde{\mathcal{L}}^{n+j} g(x) \le \dfrac{D A_+}{n^{\beta-1}}$ $\forall x \in I$,

where D, D' are as in Proposition 5.2, and $A_+ = \sup\{g(x) : x \in E^{(j)}\}$ and $A_- = \inf\{g(x) : x \in E^{(j)}\}$.

Proof. Take $0 < b_- \le b_+ < 1$ such that $\dfrac{b_+}{b_-} = \dfrac{A_+}{A_-}$. Let $v = \left(\dfrac{\beta}{s}\right)^{\beta}$ be the point given in Proposition 5.2.

For each $j > 0$, consider the function $g_j(x) = \tilde{\mathcal{L}}^j g(x)$. Since $f^j : E^{(j)} \to I$ is a one to one map,

$$g_j(x) = g(x_j) \prod_{i=1}^{j} \eta(f^i x_j) \le A_+ \prod_{i=1}^{j} \eta(f^i x_j),$$

for all $x \in I$, where $x_j = f^{-j}x \cap E^{(j)}$.

Note that if $x \le y$, then

$$\frac{g(x)\eta(x)}{g(y)\eta(y)} \ge \frac{1}{1+\tilde{J}d(x,y)} \cdot \frac{1-\psi(x)}{1-\psi(y)} \ge \frac{1+\psi(y)-\psi(x)}{1+\tilde{J}d(x,y)}.$$

It is easy to see by Lemma 4.4.i), Lemma 3.5 and 2.4 that the right side is greater than or equal to 1 if x is small. It means that $g(x)\eta(x)$ and therefore $g_j(x)$ is decreasing on $[0, v]$ if $E^{(j)}$ is small enough.

By Lemma 2.2, the length of $E^{(j)}$ is between $\left(\frac{\beta_-}{r+j}\right)^\beta$ and $\left(\frac{\beta_+}{r+j}\right)^\beta$ for some $r > 0$. So if j is large enough, then by (c), $g(y) \le 2g(x)$ for any $x, y \in E^{(j)}$. Hence by (b) and Lemma 4.2, we have

$$\frac{(r+j)^{\beta-1}}{2a\beta_+^{\beta-1}} \le \frac{1}{2\mu E^{(j)}} \le A_- \le A_+ \le \frac{2}{\mu E^{(j)}} \le \frac{2(r+j)^{\beta-1}}{a'\beta_-^{\beta-1}}. \tag{7.3}$$

On the other hand, by Lemma 4.7, $\prod_{i=1}^{j} \eta(v_i) \le \left(\frac{s}{s+j}\right)^\beta$. So if j is large enough, then $g(v) \le b_+$ and therefore $g(x) \le b_+ \ \ \forall x > v$.

Now we see that g_j satisfies all conditions in Proposition 5.2, with $j = k$. Therefore the results of the lemma follow. $\square$

Lemma 7.3. *There exist $C > 0$ and $l > 0$ such that for any $m \ge 0$, if $E \in \xi_m$, then for all $n > 0$,*

$$|\mu E - \tilde{\mathcal{L}}^{n+m+l}\chi_E(x)| \le \frac{Cm^{\beta-1}}{n^{\beta-1}}\mu E \qquad \forall x \in I\backslash I_0.$$

Proof. Note that $f^m : E \to I$ is a one to one map and $f^{m-1}E = I_q$ for some q.

First we consider the case $f^{m-1}E = I_q \ne I_0$. Put

$$g(x) = \frac{1}{\mu E}\tilde{\mathcal{L}}^m\chi_E(x) = \frac{1}{\mu E}\prod_{i=1}^{m}\eta(f^i x_m),$$

where $x_m = f^{-m}x \cap E$. By the remark after Lemma 4.8, we know $g(y) \le g(x)\big(1 + \tilde{J}d(x,y)\big)$ for any $x \in I$, $y \in B(x, \bar{\rho})$. Since $\mu(g) = 1$, by similar arguments as in the proof of Lemma 3.2 we know that g is bounded and the bounds is independent of m and E provided $f^{m-1}E \ne I_0$. Consequently, g is a Lipschitz function and the Lipschitz constant is independent of m and E. So by Proposition 7.1, we have

$$\big|\mu E - \tilde{\mathcal{L}}^{n+m}\chi_E(x)\big| \le \frac{C}{n^{\beta-1}}\mu E \le \frac{Cm^{\beta-1}}{n^{\beta-1}}\mu E \qquad \forall x \in I\backslash I_0$$

for all $n > 0$, where $C \ge B\epsilon^{-1}$.

Secondly, we consider the case that there exists $l \le j \le m$ such that $f^{m-j}E = E^{(j)} \subset I_0$, where l is as in Lemma 7.2. We may assume that j is the largest number with this property. Take

$$g(x) = \frac{1}{\mu E}\tilde{\mathcal{L}}^{m-j}\chi_E(x) = \frac{1}{\mu E}\prod_{i=1}^{m-j}\eta(f^i x_{m-j}),$$

where $x_{m-j} = f^{-m+j}x \cap E$. Clearly $g(x)$ satisfies all requirements in Lemma 7.2. So we get that for all $n > 0$

$$1 - \frac{1}{\mu E}\tilde{\mathcal{L}}^{n+m}\chi_E(x) = 1 - \tilde{\mathcal{L}}^{n+j}g(x) \le \frac{DA_+}{n^{\beta-1}} \qquad \forall x \in I\backslash I_0.$$

Recall (7.3), and note that r only depends on f. We may assume $l > r$. Since $j \le m$, we have $A_+ \le \dfrac{2^\beta m^{\beta-1}}{a'\beta_-^{\beta-1}}$. So the result follows with $C \ge \dfrac{2^\beta D}{a'\beta_-^{\beta-1}}$.

Lastly, we consider the case that $f^{m-j}E = E^{(j)} \subset I_0$ hold only for $j < l$. We take a partition $E = E_{l-j}\bigcup\big(\cup_{i=1}^{l-j-1}\cup_{q=1}^{Q} E_{i,q}\big)$ such that $f^{m+i-1}E_{i,q} = I_q$ and $f^{m+l-j-1}E_{l-j} = I_0$. For each $E_{i,q}$, we use the argument similar to the first case for the function $g(x) = (\mu E_{i,q})^{-1}\tilde{\mathcal{L}}^{m+i}\chi_{E_{i,q}}(x)$ to get that for all $n > 0$,

$$\big|\mu E_{i,q} - \tilde{\mathcal{L}}^{n+m+i}\chi_{E_{i,q}}(x)\big| \le \frac{Cm^{\beta-1}}{n^{\beta-1}}\mu E_{i,q} \qquad \forall x \in I\backslash I_0. \tag{7.4}$$

Also, we have $f^{m-j}E_{l-j} = E^{(l)}$. So by taking $g(x) = (\mu E_{l-j})^{-1}\tilde{\mathcal{L}}^{m-j}\chi_{E_{l-j}}(x)$, the same reasons as in the second case imply that for all $n > 0$,

$$\mu E_{l-j} - \tilde{\mathcal{L}}^{n+m+l-j}\chi_{E_{l-j}}(x) \le \frac{Cm^{\beta-1}}{n^{\beta-1}}\mu E_{l-j} \qquad \forall x \in I\backslash I_0. \tag{7.5}$$

Since $i \le l$ and $l-j \le l$, (7.4) and (7.5) still hold if we use $\tilde{\mathcal{L}}^{n+m+l}$ instead of $\tilde{\mathcal{L}}^{n+m+i}$ and $\tilde{\mathcal{L}}^{n+m+l-j}$ respectively. Hence the result follows if we take summation. □

Proof of Corollary of Theorem B.
Use (7.2) and take $F = \chi_{E'}$ and $G = \chi_E$, we get

$$\mu(f^{-n-m}E' \cap E) - \mu E' \cdot \mu E = \mu\big(\chi_{E'} \cdot (\tilde{\mathcal{L}}^{n+m}\chi_E - \mu E)\big).$$

Since $E' \subset I\backslash I_0$,

$$\begin{aligned}&\mu E' \cdot \min_{x\in I\backslash I_0}\big(\mu E - \tilde{\mathcal{L}}^{n+m}\chi_E(x)\big)\\ \le&\mu\big(\chi_{E'} \cdot (\mu E - \tilde{\mathcal{L}}^{n+m}\chi_E)\big) \le \mu E' \cdot \max_{x\in I\backslash I_0}\big(\mu E - \tilde{\mathcal{L}}^{n+m}\chi_E(x)\big).\end{aligned}$$

Therefore the first inequality follows from Lemma 7.3 with $n+l$ replaced by n. For the second one, we take $g(x) = (\mu E)^{-1}\chi_E(x)$ and then apply Lemma 7.2.i) with $j = m$ to get

$$\mu E - \tilde{\mathcal{L}}^{n+m}\chi_E(x) \geq \frac{D' A_-}{(n+m)^{\beta-1}}\mu E \qquad \forall x \in I \backslash I_0$$

for all $n > 0$. By (7.3), $A_- \geq (2a)^{-1}\beta_+^{1-\beta} m^{\beta-1}$. So we get the inequality by taking $C' \leq (2a)^{-1}\beta_+^{1-\beta} \cdot D'$. $\square$

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