EVERY COMPACT MANIFOLD CARRIES A HYPERBOLIC ERGODIC FLOW

HUYI HU, YAKOV PESIN, ANNA TALITSKAYA

Dedicated to Anatole Katok on the occasion of his 60th birthday.

ABSTRACT. We show that every compact smooth Riemannian manifold M of $\dim M \geq 3$, admits a volume-preserving Bernoulli flow with non-zero Lyapunov exponents except for the Lyapunov exponent along the direction of the flow.

1. INTRODUCTION

In [DP], Dolgopyat and Pesin obtained an affirmative solution to the long-standing problem of whether a compact smooth Riemannian manifold admits a volume-preserving ergodic (Bernoulli) diffeomorphism with non-zero Lyapunov exponents. In this paper we discuss a continuous time version of this problem and we prove the following result.

Theorem 1. *Given a compact smooth Riemannian manifold M of* $\dim M \geq 3$*, there exists a C^∞ flow f^t such that for each $t \neq 0$,*

(1) *f^t preserves the Riemannian volume μ on M;*
(2) *f^t has non-zero Lyapunov exponents (except for the exponent along the flow direction) at m-almost every point $x \in M$;*
(3) *f^t is a Bernoulli diffeomorphism.*

Acknowledgment. The final version of this paper was written when Ya. P. was visiting Research Institute for Mathematical Science (RIMS) at Kyoto University. Ya. P. wish to thank RIMS for hospitality.

2. CONSTRUCTION OF DIFFEOMORPHISMS BY KATOK AND BRIN

In our construction of the flow we use a special diffeomorphism of the two dimensional unit disc $\mathcal{D}^2$ constructed by Katok in [K]. We summarise the description and properties of this diffeomorphism in the following proposition.

Proposition 2. *There exists a C^∞ diffeomorphism $g : \mathcal{D}^2 \to \mathcal{D}^2$ with the following properties:*

(1) *g preserves area on $\mathcal{D}^2$;*
(2) *g has non-zero Lyapunov exponents almost everywhere;*
(3) *g is a Bernoulli map;*

Key words and phrases. Hyperbolic measure, Lyapunov exponents, stable ergodicity, accessibility.

H. H. was partially supported by NSF grants DMS-0196234 and DMS-0240097; Ya. P. and A. T. were partially supported by the National Science Foundation grant DMS-0088971 and U.S.-Mexico Collaborative Research grant 0104675.

(4) $d^k(g - \mathrm{id})|\partial\mathcal{D}^2 = 0$ *for any* $k \geq 0$, *i.e., on the boundary of the disk* g *is the identity map and has all its derivatives zero.*

Let us sketch the construction of the diffeomorphism g. We begin with the automorphism g_0 of the torus $\mathcal{T}^2$ given by the matrix $\begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$. The map g_0 has four fixed points

$$q_1 = (0,0)\,, \quad q_2 = \left(\frac{1}{2}, 0\right), \quad q_3 = \left(0, \frac{1}{2}\right), \quad q_4 = \left(\frac{1}{2}, \frac{1}{2}\right).$$

In a small neighborhood $D_r^i = \{(s_1, s_2) : s_1^2 + s_2^2 \leq r\}$ of each q_i, $0 < r < 1$, the map g_0 is the time-1 map of the flow given by

$$\dot{s}_1 = -(\log\alpha)s_1, \quad \dot{s}_2 = (\log\alpha)s_2,$$

where $\alpha > 1$ is the larger eigenvalue of g_0 and $\{s_1, s_2\}$ is the coordinate system in a neighborhoods of each q_i generated by the eigenvectors of g_0.

Then we consider the map g_1 that is conjugate to g_0 via a conjugacy ϕ_0 that slows down the motion near q_i. More precisely, g_1 is the time-1 map of the flow given by

$$\dot{s}_1 = -(\log\alpha)s_1\psi(s_1^2 + s_2^2), \quad \dot{s}_2 = (\log\alpha)s_2\psi(s_1^2 + s_2^2) \tag{2.1}$$

in D_r^i, and $g_1 = g_0$ otherwise, where ψ is a C^∞ function except at zero and such that $\psi(0) = 0$, $\psi(\xi) \geq 0$ for $\xi \geq 0$, $\psi(\xi) = 1$ for $\xi \geq r$ and

$$\int_0^r \sqrt{\frac{1}{\psi(\xi)}}\, d\xi < \infty.$$

The map g_1 preserves a probability measure $d\nu = \kappa_0^{-1}\kappa\, dm$, where m is area and the density κ is a positive C^∞ function defined by the formula

$$\kappa(s_1, s_2) = \begin{cases} (\psi({s_1}^2 + {s_2}^2))^{-1} & \text{if } (s_1, s_2) \in D_r^i. \\ 1 & \text{otherwise.} \end{cases}$$

Here

$$\kappa_0 = \int_{\mathcal{T}^2} \kappa\, dm.$$

Note that κ is infinite at q_i.

Define the map ϕ_1 by the formula

$$\phi_1(s_1, s_2) = \frac{1}{\sqrt{\kappa_0({s_1}^2 + {s_2}^2)}} \left(\int_0^{{s_1}^2 + {s_2}^2} \frac{du}{\psi(u)}\right)^{1/2} (s_1, s_2)$$

near each q_i and then extend the map to $\mathcal{T}^2$ in such a way that ϕ_1 is C^∞, commutes with the involution $J(t_1, t_2) = (1 - t_1, 1 - t_2)$ on $\mathcal{T}^2$, and satisfies $(\phi_1)_*\nu = m$. Hence, $g_2 = \phi_1 \circ g_1 \circ \phi_1^{-1}$ is a C^∞ area preserving map.

Let $\phi_2 \colon \mathcal{T}^2 \to S^2$ be a double branched covering that satisfies $\phi_2 \circ J = \phi_2$, $(\phi_1)_*m = m$, and C^∞ everywhere except for the points q_i, where it branches and near q_i,

$$\phi_2(s_1, s_2) = \frac{1}{\sqrt{{s_1}^2 + {s_2}^2}}(s_1^2 - s_2^2,\ 2s_1s_2).$$

The map $g_3 = \phi_2 \circ g_2 \circ \phi_2^{-1}$ is a C^∞ diffeomorphism of the sphere $\mathcal{S}^2$.

Finally, let ϕ_3 be a C^∞ map that blows up the point q_4 into a circle and makes $g = \phi_3 \circ g_3 \circ \phi_3^{-1}$ to be the desired map of the disk. We refer the reader to [K] (see also [DP]) for more details.

Note that the map g_1 preserves measure ν and the maps g_2, g_3 and g preserve area. The desired result follows.

We call the map g *Katok's map*. We need some additional properties of this map.

Since g has non-zero Lyapunov exponents, for almost every x there are global stable and unstable manifolds, $\mathcal{W}_g^s(x)$ and $\mathcal{W}_g^u(x)$, at x.

Let $Q = \{q_1, q_2, q_3\} \cup \partial\mathcal{D}^2$ be the *discontinuity set* of g.

Proposition 3. *The following properties hold:*

1. *periodic points of g are everywhere dense;*
2. *g possesses two one-dimensional continuous foliations which are extensions of the stable and unstable global foliations $W_g^s(x)$ and $W_g^u(x)$; we will use the same notations for these foliations;*
3. *there exist neighborhoods $U \subset U_1$ of $\partial\mathcal{D}^2$ and a vector field V in U_1 which generates an area-preserving flow $g^t\colon U \to \mathcal{D}^2$, $-2 < t < 2$ for which $g|U = g^1$.*

Proof. Note that the map g_1 is topologically conjugate to g_0 and Statement 1 follows. Statements 2 is proved in [K] (see Lemma 4.1). We prove the last statement. By construction, near each point q_i, the map g_1 is the time-1 map of a vector field V_1. Moreover, near each q_i we have $V_1(-x) = -V_1(x)$ for any x, see (2.1). It follows that the maps g_2 and g_3 near q_i are the time-1 maps of the vector fields given by $V_2 = d\phi_1 \circ V_1 \circ \phi_1^{-1}$ and $V_3 = d\phi_2 \circ V_2 \circ \phi_2^{-1}$ respectively. Here we should stress that V_3 is well defined even though ϕ_2 is a two to one map. In fact, near q_i we have $\phi_1(-x) = -\phi_1(x)$ and $\phi_2(-x) = \phi_2(x)$. This gives $V_2(-x) = -V_2(x)$ and therefore, for any y near q_i, $\phi_2^{-1}(y)$ has two preimages x and $-x$ at which

$$(d\phi_2)_{-x}(V_2(-x)) = (d\phi_2)_{-x}(-V_2(x)) = (d\phi_2)_x(V_2(x)).$$

Now we see that g is the time-1 map of the vector field $V = d\phi_3 \circ V_2 \circ \phi_3^{-1}$. □

The next result shows that the map g is diffeotopic to the identity map.

Proposition 4. *There exists a map $G : \mathcal{D}^2 \times [0,1] \to \mathcal{D}^2$ with the following properties:*

1. *$G(x,t)$ is C^∞ in (x,t);*
2. *$G(\cdot,0) = \mathrm{id}$ and $G(\cdot,1) = g$;*
3. *$G(x,t) = g^t(x)$ for any $x \in U$ and $t \in [0,1]$, where $g^t(x)$ is the flow in Proposition 3;*
4. *for any $t \in [0,1]$ the map $G(\cdot,t) : \mathcal{D}^2 \to \mathcal{D}^2$ is an area-preserving diffeomorphism;*
5. *$d^kG(x,1) = d^kG(g(x),0)$ for any $k \geq 0$.*

Proof. Recall that in the neighborhood U of the boundary of $\mathcal{D}^2$ the map g is the time-1 map of the flow generated by the vector field V. We extend V to a smooth vector field $\widehat{V}$ on the whole $\mathcal{D}^2$, and let $\widehat{g}^t$ be the flow generated by $\widehat{V}$. Note that $g|U = \widehat{g}^1|U$. We need the following result of Smale (see [S], Theorem B):

Lemma 5. *Let $\mathcal{A}$ be the space of C^∞ diffeomorphisms of the unit square which are equal to the identity in some neighborhood of the boundary. Endow $\mathcal{A}$ with the C^r topology, $1 < r \leq \infty$. Then $\mathcal{A}$ is contractible to a point.*

The statement also holds if the unit square is replaced by the unit disk. Applying the result to the diffeomorphism $g \circ \widehat{g}^{-1}$, which is equal to the identity on U, we obtain a homotopy $\widetilde{G} : \mathcal{D}^2 \times [0,1] \to \mathcal{D}^2$ such that $\widetilde{G}(\cdot,0) = \mathrm{id}\,|\mathcal{D}^2$ and $\widetilde{G}(\cdot,1) = \widehat{g}^{-1} \circ g$. Moreover, $\widetilde{G}$ is C^∞ in (x,t), i.e., $\widetilde{G}$ is a diffeotopy in $\mathcal{A}$ (see [S], Theorem 4). Therefore, for each $t \in [0,1]$, there is a neighborhood U_t of $\partial\mathcal{D}^2$ such that $\widetilde{G}(\cdot,t)|U_t = \mathrm{id}\,|U_t$. One can show that the set

$$U = \mathrm{int} \bigcap_{t\in[0,1]} U_t$$

is not empty and is a neighborhood of $\partial\mathcal{D}^2$. Denote $\widetilde{g}^t = \widetilde{G}(\cdot,t)$. It follows that the map $G(\cdot,t) = \widetilde{g}^t \circ \widehat{g}^t$ satisfies Statements 1- 3 of the proposition. To prove Statements 4 and 5, we need the following lemma.

Lemma 6. *Let $\{\Omega_0^t\}$ and $\{\Omega_1^t\}$ be two families of volume forms on $\mathcal{D}^2$ that are C^∞ in (x,t). Assume that $\Omega_0^t|U = \Omega_1^t|U$ for any t and $\Omega_0^t = \Omega_1^t$ for $t \in [0,\varepsilon) \cup (1-\varepsilon, 1]$. Then there exists a map $\bar{G} : \mathcal{D}^2 \times [0,1] \to \mathcal{D}^2$ with the following properties:*

1. *$\bar{G}(x,t)$ is C^∞ in (x,t);*
2. *$\bar{G}(\cdot,0) = \bar{G}(\cdot,1) = \mathrm{id}$;*
3. *for any $t \in [0,1]$ the map $\bar{G}(\cdot,t) : \mathcal{D}^2 \to \mathcal{D}^2$ is a diffeomorphism with $\bar{G}(\cdot,t)^*\Omega_1^t = \Omega_0^t$;*
4. *$\bar{G}(x,t) = x$ for any $t \in [0,1]$ and x in some neighborhood $U' \subset U$ of $\partial\mathcal{D}^2$.*

Proof. The argument is a modification of the proof of Moser's theorem ([M]). We follow here the approach in [KH] (see Theorem 5.1.27). Let $\Omega^t = \Omega_1^t - \Omega_0^t$ and $\Omega_s^t = \Omega_0^t + s\Omega^t$ for $s \in [0,1]$. We know that $\Omega^t|U = 0$. We construct a family of one forms $\mathcal{H}^t$ such that $\mathcal{H}^t$ is C^∞ in (x,t), $d\mathcal{H}^t = \Omega^t$ and $\mathcal{H}^t|U' = 0$ for some neighborhood $U' \subset U$ of $\partial\mathcal{D}^2$. Consider a Euclidean coordinate system (x_1,x_2) such that $\mathcal{D}^2 = \{(x_1,x_2) : x_1^2 + x_2^2 \le 1\}$. We have

$$\Omega^t = \rho^t(x)dx_1 \wedge dx_2$$

and

$$\Omega_s^t = \rho_s^t(x)dx_1 \wedge dx_2$$

with $\rho_s^t > 0$. Note that $\rho^t|_U = 0$ and $\int \rho^t\,dx = 0$ for any $t \in [0,1]$. Given a C^∞ function $\theta^t = \theta^t(x)$, $x \in \mathcal{D}^2$, $t \in [0,1]$, let

$$E_1(\theta^t)(a) = \int_{\mathcal{D}^2\cap\{x_2=a\}} \theta^t(x_1,a)dx_1$$

and

$$E_2(\theta^t)(a) = \int_{\mathcal{D}^2\cap\{x_1=a\}} \theta^t(a,x_2)dx_2$$

be the expectation of θ^t along the lines $x_2 =$ constant and $x_1 =$ constant. We choose the function θ^t such that

$$E_1(\theta^t) = E_1(\rho^t), \quad E_2(\theta^t) = 0,$$

and $\theta^t|U' = 0$ where $U' \subset U$ is a neighborhood of $\partial\mathcal{D}^2$. Such θ^t exists. Indeed, choose any positive C^∞ function δ such that $\int \delta dx_1 = 1$, the support $\mathrm{supp}\,\delta \subset (-\varepsilon,\varepsilon)$ for some small $\varepsilon > 0$, and then set $\theta^t(x_1,x_2) = \delta(x_1)E_1(\rho^t)(x_2)$. Now let

$$\begin{aligned} a^t(x_1,x_2) &= -\int_{\mathcal{D}^2\cap\{(x_1,y):y<x_2\}} \theta^t(x_1,y)\,dy, \\ b^t(x_1,x_2) &= \int_{\mathcal{D}^2\cap\{(y,x_2):y<x_1\}} \left[\rho^t(y,x_2) - \theta^t(y,x_2)\right] dy. \end{aligned}$$

The 1-forms
$$\mathcal{H}^t(x_1, x_2) = a^t(x_1, x_2)dx_1 + b^t(x_1, x_2)dx_2$$
satisfy the desired requirements.

For each $s \in [0, 1]$, consider the vector field
$$V_s^t = -\frac{b^t}{\rho_s^t}\frac{\partial}{\partial x_1} + \frac{a^t}{\rho_s^t}\frac{\partial}{\partial x_2}.$$
It is well defined since $\rho_s^t > 0$ on $\mathcal{D}^2$ and it is C^∞ in (x, t, s). We also have $V_s^t|U' = 0$ and $\Omega_s^t(V_s^t, \cdot) = -\mathcal{H}^t$. Let $(\bar{G}^t)^s \colon \mathcal{D}^2 \to \mathcal{D}^2$ be the solution of the differential equation $\frac{dx}{ds} = V_s^t(x)$ satisfying the initial condition $(\bar{G}^t)^0 = \text{id}$. We have that
$$(d\bar{G}^t)^{1*}\Omega_1^t = (d\bar{G}^t)^{0*}\Omega_0^t = \Omega_0^t$$
(see [KH], Theorem 5.1.27 for more details). So $\bar{G}^t = (\bar{G}^t)^1$ is the desired map. $\square$

We proceed with the proof of the proposition. Consider the map $\bar{G}$ as in the lemma with $\Omega_0^t = dx_1 \wedge dx_2$ on $\mathcal{D}^2$ for any $t \in [0, 1]$ and $\Omega_1^t = d\tilde{g}^{t*}d\hat{g}^{t*}\Omega_0^t$. Let $\bar{g}^t = \bar{G}(\cdot, t)$. Then the map $G : \mathcal{D}^2 \times [0, 1] \to \mathcal{D}^2$,
$$G(x, t) = \bar{g}^t \circ \hat{g}^t \circ \tilde{g}^t$$
satisfies Statements 1 – 4 of the proposition. If necessary one can change the map $\tilde{G}(\cdot, t)$ in a small neighborhood of the set $\mathcal{D}^2 \times 0$ and $\mathcal{D}^2 \times 1$ so that it will also satisfy Statement 5. For example, one can choose $\widetilde{G}$ in such a way that
$$\widetilde{G}\big(\widehat{G}(x, t), t\big) = \widetilde{G}\big(\widehat{G}(x, 0), 0\big),\ t \in [0, \varepsilon), \quad \widetilde{G}\big(\widehat{G}(x, t), t\big) = \widetilde{G}\big(\widehat{G}(x, 1), 1\big),\ t \in (1 - \varepsilon, 1].$$
Hence, $\hat{g}^t \circ \tilde{g}^t$ is area-preserving for any $t \in [0, \varepsilon) \cup (1-\varepsilon, 1]$. Therefore, by Lemma 6, $\Omega_1^t = \Omega_0^t$ and $\bar{G}(\cdot, t) = \text{id}$ for all $t \in [0, \varepsilon) \cup (1 - \varepsilon, 1]$, because in this case $\Omega^t = 0$, $\rho^t = 0$, $a^t = b^t = 0$ and $V_s^t = 0$ for every s. $\square$

We also need Brin's construction from [B2]. Given $n \geq 5$, let $A \colon \mathcal{T}^{n-3} \to \mathcal{T}^{n-3}$ be a hyperbolic automorphism of the $(n-3)$-dimensional torus and $h^t \colon L \to L$ the suspension flow over A with the roof function $H = 1$, $y \in \mathcal{T}^{n-3}$. The suspension manifold L is diffeomorphic to $\mathcal{T}^{n-3} \times [0, 1]/\sim$, where $\sim$ is the identification $(y, 1) = (Ay, 0)$. The flow h^t preserves volume on L and one can choose A so that L can be embedded into $\mathcal{R}^{n-1} \times \mathcal{S}^1$ with trivial normal bundle.

3. Proof of the theorem: the case $\dim M \geq 5$

Consider the map
$$R = g \times A : \mathcal{D}^2 \times \mathcal{T}^{n-3} \to \mathcal{D}^2 \times \mathcal{T}^{n-3},$$
where $g \colon \mathcal{D}^2 \to \mathcal{D}^2$ is Katok's diffeomorphism and $A \colon \mathcal{T}^{n-3} \to \mathcal{T}^{n-3}$ is the automorphism from Brin's construction. Consider the suspension flow over R with roof function $H = 1$ and the suspension manifold $K = \mathcal{D}^2 \times \mathcal{T}^{n-3} \times [0, 1]/\sim$, where $\sim$ is the identification $(x, y, 1) = (g(x), A(y), 0)$. In other words, K is the manifold $\mathcal{D}^2 \times \mathcal{T}^{n-3} \times [0, 1]$ where the points $(x, y, 1)$ are identified with the points $(g(x), A(y), 0)$. Denote by Z the vector field of the suspension flow and by φ_Z^t the suspension flow itself.

For each point $(x, y, t) \in K$ consider the coordinate system
$$(x_1, x_2, y_1, \dots, y_{n-3}, t) = (x, y, t) \tag{3.2}$$

in its neighborhood where $x = (x_1, x_2) \in \mathcal{D}^2$, $y = (y_1, \dots y_{n-3}) \in \mathcal{T}^{n-3}$ and $t \in [0, 1]$. In this coordinate system, $Z = (0, 0, 1)$.

Set $N = \mathcal{D}^2 \times L$, where L is the suspension manifold in Brin's construction (see the previous section). Write $N = \mathcal{D}^2 \times (\mathcal{T}^{n-3} \times [0,1]/\sim)$ where $\sim$ is the identification $(y, 1) = (A(y), 0)$ for all $y \in \mathcal{T}^{n-3}$.

Consider the map $F \colon K \to N$ given by

$$F(x, y, t) = (G(x, t),\, y,\, t),$$

where $G : \mathcal{D}^2 \times [0, 1] \to \mathcal{D}^2$ is the diffeotopy constructed in Proposition 4. We have

$$F(x, y, 1) = (g(x), y, 1) = (g(x), A(y), 0) = F(g(x), A(y), 0).$$

Therefore, the map F is well-defined. It is easy to see that F preserves volume, is one-to-one and continuous. Hence, it is a homeomorphism. Formal differentiation yields

$$dF(x, y, t) = \begin{pmatrix} G_x(x,t) & 0 & G_t(x,t) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3}$$

It follows from Statement 5 of Proposition 4 that for every $k \geq 1$,

$$d^k F(x, y, 1) = d^k F(g(x), Ay, 0)$$

and hence, F is a C^∞ diffeomorphism.

Consider the vector field $Y = dFZ$ on N and let φ^t_Y be the corresponding flow. In the coordinate system (3.2), we have

$$Y(G(x,t), y, t) = \Big(\frac{\partial G}{\partial t}(x,t), 0, 1\Big), \qquad (x, y, t) \in K. \tag{3.4}$$

The vector field Y is divergence free since it is the image of the divergence free vector field Z under the volume-preserving map F.

Let us choose a C^∞ function $\alpha : \mathcal{D}^2 \to [0, 1]$ such that

(A1) α and all its partial derivatives of any order are equal to zero on $\partial\mathcal{D}^2$;
(A2) $\alpha(x) > 0$ for $x \in \operatorname{int}\mathcal{D}^2$, $\alpha(x) = 1$ for $x \in \mathcal{D}^2 \setminus U$.

Define the vector field X on N by

$$X(G(x,t), y, t) = \Big(\frac{\partial G}{\partial t}(x,t), 0, \alpha(G(x,t))\Big), \qquad (x, y, t) \in K. \tag{3.5}$$

Note that by Statement 3 of Proposition 4, $\dfrac{\partial G}{\partial t}(x,t) = V(G(x,t))$ for $x \in U$. Therefore, Equalities (3.4) and (3.5) imply that for $(x, y, t) \in N$ with $x \in U$,

$$Y(x, y, t) = (V(x), 0, 1), \qquad X(x, y, t) = (V(x), 0, \alpha(x)). \tag{3.6}$$

Let $\varphi^t = \varphi^t_X$ be the flow on N generated by the vector field X. It is easy to see that X is divergence free and hence, φ^t is volume-preserving.

Lemma 7. *All but one Lyapunov exponents of the flow φ^t_X are non-zero almost everywhere.*

Proof. We begin with a construction of a map which is similar to Katok's map: it has the same topological and hyperbolic properties but does not preserve area. It is better adopted to the flow. We will also use this map in the proof of the ergodicity of the flow.

We assume that the function $\alpha(x)$ is chosen such that the following additional condition holds:

(A3) $\alpha(x)^{-1}V(x) \to 0$ as $x \to \partial\mathcal{D}^2$, where V is the vector field defined in Statement 3 of Proposition 3.

Consider the map $g^* : \mathcal{D}^2 \to \mathcal{D}^2$ such that $g^* = g$ on $\mathcal{D}^2 \backslash U$ and g^* is the time-1 map of the flow ${g^*}^t$ generated by the vector field $V^* = \alpha^{-1}V$ on U_1 (see Statement 3 of Proposition 3). By $(A1) - (A3)$, the map g^* is well defined and is a diffeomorphism. It also satisfies Statements 2 – 4 of Propositions 2 and 3. Note that the map g^* preserves a measure μ^* which is absolutely continuous with respect to area with positive density $\rho^*(x)$; the latter is unbounded as x approaches $\partial\mathcal{D}^2$.

We now proceed as before replacing g by g^*. Namely, define $G^* : \mathcal{D}^2 \times [0,1] \to \mathcal{D}^2$ by $G^*(x,t) = G(x,t)$ if $x \in X \setminus U$, and $G^*(x,t) = {g^*}^t(x)$ otherwise. Let $\phi^t_{Z^*}$ be the suspension flow over $g^* \times A$ with the suspension manifold $K^* = X \times \mathcal{T}^{n-3} \times [0,1]/\sim$, where $\sim$ is the identification $(x,y,1) = (g^*(x), A(y), 0)$ and Z^* is the vector field of the suspension flow. Define the map $F^* : K^* \to N$ by

$$F^*(x,y,t) = (G^*(x,t), y, t)$$

and let $Y^* = dF^* Z^*$. We have for $x \in U$,

$$\frac{\partial G^*}{\partial t}(x,t) = \alpha(g^t(x))^{-1}V(g^t(x))$$

and hence,

$$Y^*(G^*(x,t),y,t)) = (\alpha({g^*}^t(x))^{-1}V({g^*}^t(x)), 0, t)$$

or equivalently,

$$Y^*(x,y,t) = (\alpha(x)^{-1}V(x), 0, t).$$

Since $Y^*(x,y,t) = Y(x,y,t)$ for $x \notin U$, we obtain by (3.6) that $X = \alpha Y^*$.

Define the vector field $\tilde{Z}$ on K^* by

$$\tilde{Z}(x,y,t) = (dF^*)^{-1}X(F^*(x,y,t)) \qquad (x,y,t) \in K^*.$$

It is easy to see that $\phi^t_X = F^* \circ \phi^t_{\tilde{Z}} \circ {F^*}^{-1}$ and the vector fields Z^* and $\tilde{Z}$ have the same orbits. In other words, there is a function $\beta\colon K^* \to \mathcal{R}^+$ such that for every $(x,y,t) \in K^*$,

(B1) $\tilde{Z}(x,y,t) = \beta(x,y,t)Z^*(x,y,t)$;
(B2) $\beta(x,y,t) = 0$ if $x \in \partial\mathcal{D}^2$;
(B3) $\beta(x,y,t) = 1$ if $x \notin U$.

By construction, the flow $\phi^t_{Z^*}$ has non-zero Lyapunov exponents almost everywhere. Denote by $E^s_{Z^*}(x,y,t)$, $E^u_{Z^*}(x,y,t)$, $E^{cs}_{Z^*}(x,y,t)$ and $E^{cu}_{Z^*}(x,y,t)$ the stable, unstable, center-stable and center-unstable invariant subspaces at the point $(x,y,t) \in K^*$ repectively. Observe that the subspaces $E^{cs}_{Z^*}(x,y,t)$ and $E^{cu}_{Z^*}(x,y,t)$ are also invariant under the flow $\phi^t_{\tilde{Z}}$. Chose a point (x_0,y_0,t_0) and a vector $v \in E^u_{Z^*}(x_0,y_0,t_0)$. Note that for almost every (x_0,y_0,t_0) (with respect to the Riemannian volume) the proportion of time the trajectory $\{\phi^t_{\tilde{Z}}(x_0,y_0,t_0)\}$ spends in the set $\{(x,y,t)\colon x \notin U\}$ is strictly positive. It follows that the Lyapunov exponent $\chi(v)$ at (x_0,y_0,t_0) with respect to the flow $\phi^t_{\tilde{Z}}$ is positive. □

Almost every point $(x_0,y_0,t_0) \in N$ has stable, unstable, center-stable and center-unstable global manifolds $W^s_X(x_0,y_0,t_0)$, $W^u_X(x_0,y_0,t_0)$, $W^{cs}_X(x_0,y_0,t_0)$ and $W^{cu}_X(x_0,y_0,t_0)$ with respect to the flow φ^t_X. Similarly, almost every point $(x_0,y_0,t_0) \in K^*$ has stable, unstable, center-stable and center-unstable global manifolds $W^s_{\tilde{Z}}(x_0,y_0,t_0)$, $W^u_{\tilde{Z}}(x_0,y_0,t_0)$, $W^{cs}_{\tilde{Z}}(x_0,y_0,t_0)$ and $W^{cu}_{\tilde{Z}}(x_0,y_0,t_0)$ with respect to the flow $\varphi^t_{\tilde{Z}}$. By Proposition 2 these foliations can be extended to foliations which are continuous everywhere except for the discontinuity set $Q \times \mathcal{T}^{n-3} \times [0,1]/\sim$. We will use the same notations for these foliations. Observe that

$$\pi_x(W^{cs}_{\tilde{Z}}(x_0,y_0,t_0)) = W^s_{g^*}(x_0), \quad \pi_x(W^{cu}_{\tilde{Z}}(x_0,y_0,t_0)) = W^u_{g^*}(x_0) \tag{3.7}$$

and

$$\pi_y(W^{cs}_{\tilde{Z}}(x_0,y_0,t_0)) = W^s_A(y_0), \quad \pi_y(W^{cu}_{\tilde{Z}}(x_0,y_0,t_0)) = W^u_A(y_0), \tag{3.8}$$

where $\pi_x\colon K^* \to \mathcal{D}^2$ and $\pi_y\colon K^* \to \mathcal{T}^{n-3}$ are natural projections.

We say that two points $z, z' \in N$ are *accessible* if there are points $z = z_0, z_1, \dots, z_{\ell-1}, z_\ell = z'$, $z_i \in N$ such that $z_i \in W^u_X(z_{i-1})$ or $z_i \in W^s_X(z_{i-1})$ for $i = 1, \dots, \ell$. The collection of points $z_0, z_1, \dots, z_\ell$ is called a *path* connecting z and z' and is denoted by $[z,z'] = [z_0, z_1, \dots, z_\ell]$. Accessibility is an equivalence relation. We say that the time-t map of the flow φ^t_X has *accessibility property* if the partition into accessibility classes is trivial (i.e. any two points z, z' are accessible) and to have *essential accessibility property* if the partition into accessibility classes is ergodic (i.e. a measurable union of equivalence classes must have zero or full measure). Similarly, one defines (essential) accessibility of the time-t map of the flow $\varphi^t_{\tilde{Z}}$ using its global stable and unstable foliations.

Lemma 8. *For every t the time-t map of the flow φ^t_X has essentially accessibility property. Moreover, for any set $\mathcal{E}$ of zero measure and almost any two points $z, z' \notin \mathcal{E}$ one can find a path $[z,z'] = [z_0, z_1, \dots, z_\ell]$ such that each $z_i \notin \mathcal{E}$.*

Proof. It suffices to establish essential accessibility property of the time-t map of the flow $\varphi^t_{\tilde{Z}}$. Note that the map g^* has essential accessibility property, indeed, any two points outside of the discontinuity set Q are accessible. Note also that the automorphism A in Brin's construction is accessible. Denote by $Q^* = \{(x,y,t) \in K^* : x \in Q, y \in \mathcal{T}^{n-3}, t \in [0,1]\}$. Fix t and consider the time-t map of the flow $\varphi^t_{\tilde{Z}}$. Relations (3.7) and (3.8) imply that for any two points (x,y) and $(x',y') \in \mathcal{D}^2 \times \mathcal{T}^{n-3}$ and any $t \in [0,1]$ the point (x,y,t) is accessible to a point (x',y',t'). In particular, any point $(x,y,t) \in K^* \setminus Q^*$ is accessible to a point in $\Pi_{p,q} = \{(p,q,t') \in K^* :\ t' \in [0,1]\}$ for some $p \in \mathcal{D}^2$ and $q \in \mathcal{T}^{n-3}$. It remains to show that any two points in $\Pi_{p,q}$ are accessible.

Let $q \in \mathcal{T}^{n-3}$ be a periodic point of the automorphism A in Brin's construction and p, p' be two periodic points of g^* such that the orbit of p and the orbit of p' except for p' itself (under g^*) lie outside the neighborhood U. Consider local stable and unstable leaves (curves) at p and p', $V^s_{g^*}(p)$, $V^u_{g^*}(p)$, $V^s_{g^*}(p')$, and $V^u_{g^*}(p')$. One can choose points p and p' such that the local unstable leaf from p to $V^u_{g^*}(p) \cap V^s_{g^*}(p')$, and the local stable and unstable leaves from p to $V^s_{g^*}(p) \cap V^u_{g^*}(p')$ also lie outside the neighborhood U.

For a point $(p,q,t) \in \Pi_{p,q}$, let $\gamma^u_p \subset W^u_{\tilde{Z}}(p,q,t)$ and $\gamma^s_p \subset W^s_{\tilde{Z}}(p,q,t)$ be the curves such that $\pi_x(\gamma^u_p) \subset V^u_{g^*}(p)$, $\pi_y(\gamma^u_p) = q$ and $\pi_x(\gamma^s_p) \subset V^s_{g^*}(p)$, $\pi_y(\gamma^s_p) = q$. Since γ^u and γ^s lie outside U and the point (p,q) is periodic, the t coordinate of a point remains unchanged when this point moves along γ^u_p from (p,q,t) toward $\gamma^u_p \cap \pi_x^{-1}(V^u_{g^*}(p) \cap V^s_{g^*}(p')) \cap \pi_y^{-1}(q)$. So does the t coordinate of a point moving along γ^s_p from (p,q,t) toward $\gamma^s_p \cap \pi_x^{-1}(V^u_{g^*}(p') \cap V^s_{g^*}(p)) \cap \pi_y^{-1}(q)$. On the other hand, for a point $(p',q,\tau) \in \Pi_{p',q}$, if we choose $\gamma^u_{p'}$ and $\gamma^s_{p'}$ in a similar way, then the third coordinate τ increases when a point moves along $\gamma^u_{p'}$ from (p,q,τ) toward $\gamma^u_p \cap \pi_x^{-1}(V^u_{g^*}(p') \cap V^s_{g^*}(p)) \cap \pi_y^{-1}(q)$ and the third coordinate τ decreases when a point moves along $\gamma^s_{p'}$ from (p,q,τ) toward $\gamma^u_p \cap \pi_x^{-1}(V^u_{g^*}(p) \cap V^s_{g^*}(p')) \cap \pi_y^{-1}(q)$ (we assume that $\alpha(p') < 1$). We can now use the argument in [DHP] (see the proof of Lemma B.4) to obtain that the point (p,q,t) is accessible to some point $(p,q,t') \in \Pi_p$ with $t' < t$, and then any two points in Π_p are accessible. This show that for every t the time-t map of the flow φ_X^t has essentially accessibility property.

Let us note that for any zero measure set in the disk and almost any two points outside this set there exists a path for Katok's map which consists of points lying outside this set. Similar statement holds for the automorphism in Brin's construction.

To complete the proof consider a set $\mathcal{E}$ of zero measure. For almost any two points $z, z' \notin \mathcal{E}$ and almost every points w and w' close to (p,q) one can find paths connecting the point z to (w,s) and z' to (w',s') for some s and s' such that every point in these paths lies outside $\mathcal{E}$. Note that the quadrilateral path from (p,q,t) to (p,q,t') in the above argument can be replaced by a nearby paths from (w,s) to (w',s') such that both (w,s) and (w',s') and all other points in the path do not belong to $\mathcal{E}$. The desired result follows. $\square$

Lemma 9. *The flow φ_X^t on N is Bernoulli.*

Proof. By results in [P3], a flow with non-zero Lyapunov exponents is Bernoulli if it is a K-flow, i.e., the Pinsker algebra $\mathcal{P}$, (the largest subalgebra for which $\phi^t|\mathcal{P}$ has zero entropy), is trivial.

Let $\mathcal{B}$ be the σ-algebra of Borel subsets in N and $\mathcal{A} \subset \mathcal{B}$ a subalgebra. Denote by $\mathrm{Sat}_0(\mathcal{A})$ the *saturation of* $\mathcal{A}$ by sets of measure zero, that is,

$$\mathrm{Sat}_0(\mathcal{A}) = \{B \in \mathcal{B} : \text{ there exists } A \in \mathcal{A} \text{ such that } \mu(A \triangle B) = 0\}$$

(where μ is volume). Let $\mathcal{S}^s$ (respectively, $\mathcal{S}^u$) be the subalgebra of Borel sets that consist of whole stable (respectively, unstable) leaves, that is, for $E \in \mathcal{S}^s$ (respectively, $E \in \mathcal{S}^u$) and $x \in E$ we have $W_X^s(x) \subset E$ (respectively, $W_X^u(x) \subset E$). By [P2] (see Theorem 2), the Pinsker algebra $\mathcal{P}$ is contained in $\mathrm{Sat}_0(\mathcal{S}^s)$. Similarly, $\mathcal{P}$ is contained in $\mathrm{Sat}_0(\mathcal{S}^u)$. Therefore, $\mathcal{P}$ is contained in $\mathrm{Sat}_0(\mathcal{S}^u) \cap \mathrm{Sat}_0(\mathcal{S}^u)$ and we wish to show that this intersection is the trivial algebra.

Let $\mathcal{A} \subset \mathrm{Sat}_0(\mathcal{S}^u) \cap \mathrm{Sat}_0(\mathcal{S}^u)$ with $\mu(\mathcal{A}) > 0$. We shall show that $\mu(\mathcal{A}) = 1$. Recall that a point $x \in N$ is a *density point* of $\mathcal{A}$ if

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap \mathcal{A})}{\mu(B(x,r))} = 1.$$

Denote by $D(\mathcal{A})$ the set of density points of $\mathcal{A}$. By the Lebesgue-Vitali theorem $D(\mathcal{A}) = \mathcal{A}$ (mod 0) and hence, it suffices to show that $\mu(D(\mathcal{A})) = 1$. Let $\mathcal{E}_1$ the complement of $D(\mathcal{A}) \cup D(N \backslash \mathcal{A})$. Applying again the Lebesgue-Vitali theorem we obtain that $\mu(\mathcal{E}_1) = 0$.

Recall that the diffeomorphism $g^* \colon \mathcal{D}^2 \to \mathcal{D}^2$ has non-zero Lyapunov exponents and results of smooth ergodic theory applies (see [BP] for relevant notions and details). Let $\mathcal{R}^\ell \subset \mathcal{D}^2$ be the regular set (of level ℓ) for g^*. The set $\bigcup_\ell \mathcal{R}^\ell$ has full measure in $\mathcal{D}^2$ and so does the set $\overline{\mathcal{R}} = \bigcup_\ell D(\mathcal{R}^\ell)$.

For $x \in \mathcal{D}^2$ and outside the discontinuity set let $\gamma_{g^*}^s(x,r)$ and $\gamma_{g^*}^u(x,r)$ be arcs in $V_{g^*}^s(x)$ and $V_{g^*}^u(x)$ centered at x of length r.

Choose two points x and x' outside the discontinuity set which are sufficiently close to each other. Consider the holonomy map $\theta : V_{g^*}^s(x) \to W_{g^*}^s(x')$ generated by the family of local leaves $V_{g^*}^u(y)$, $y \in V_{g^*}^s(x)$. Since the foliation $W_{g^*}^s$ is absolutely continuous the map θ moves the conditional measure (length) μ_x^s on $V_{g^*}^s(x)$ to a measure on $V_{g^*}^s(x')$ which is absolutely continuous with respect to $\mu_{x'}^s$. The Jacobian $\mathrm{Jac}(\theta)(y)$ is not bounded. It is however bounded if we allow y to run over the set $V^s_{g^*}(x) \cap \mathcal{R}^\ell$. More precisely, for every $x \in \mathcal{R}^\ell$ there is $J = J(\ell)$ such that

$$\mu^s_{x'}(\theta(V^s_{g^*}(x) \cap \mathcal{R}^\ell)) \le J\mu^s_x(V^s_{g^*}(x) \cap \mathcal{R}^\ell). \tag{3.9}$$

Since $x' \in D(\mathcal{R}^\ell)$ without loss of generality we may assume that for sufficiently small r,

$$\mu^s_{x'}(\gamma^s(x', 3Jr) \cap \mathcal{R}^\ell) \ge 0.9\mu^s_{x'}(\gamma^s(x', 3Jr)). \tag{3.10}$$

We claim that

$$\theta(\gamma^s(x, r)) \subset \gamma^s(x', 3Jr) \tag{3.11}$$

for all sufficiently small r. Indeed, if (3.11) does not hold then $\mu^s_{x'}(\gamma') \ge 3Jr$ where $\gamma' = \theta(\gamma)$ is the longer component of $\theta(\gamma^s(x, r)) \setminus \{0\}$ and $\gamma = \theta^{-1}(\gamma') \subset \gamma^s(x, r)$. By (3.10), we have

$$\mu^s_{x'}(\gamma^s(x', 3Jr) \cap \gamma' \cap (N \setminus \mathcal{R}^\ell)) \le 0.2\mu^s_{x'}(\gamma^s(x', 3Jr) \cap \gamma').$$

It follows that

$$\begin{aligned}\mu^s_{x'}(\theta(\gamma^s(x, r)) \cap \mathcal{R}^\ell) &\ge \mu^s_{x'}(\gamma^s(x', 3Jr) \cap \gamma' \cap \mathcal{R}^\ell) \ge 0.8\mu^s_{x'}(\gamma^s(x', 3Jr) \cap \gamma')\\ &\ge 0.8 \cdot 0.5\, \mu^s_{x'}(\gamma^s(x', 3Jr)) \ge 0.4 \cdot 3J\mu^s_{x'}(\gamma^s(x', 3Jr)) > J\mu^s_x(\gamma^s(x, r)),\end{aligned}$$

contradicting to (3.9).

Let $\mathcal{E}_2$ be the set of $(x, y, t) \in N$, where $x \in \mathcal{D}^2$ with $x \notin D(\mathcal{R}^\ell)$ for any $\ell > 0$, $y \in \mathcal{T}^{n-3}$ and $t \in \mathbb{R}$. Clearly, $\mu(\mathcal{E}_2) = 0$. Let $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$. We have $\mu(\mathcal{E}) = 0$.

Choose a point $z \in D(\mathcal{A}) \setminus \mathcal{E}$ and a point $z' \in V^u_X(z) \setminus \mathcal{E}$ (recall that $V^u_X(z)$ is a local unstable manifold at z for the flow ϕ^t_X). Consider the holonomy map $\theta : V^{cs}_X(z) \to W^{cs}_X(z')$ generated by the family of local unstable manifolds of ϕ^t_X (here $V^{cs}_X(z)$ is a local center-stable manifold at z and $W^{cs}_X(z')$ is the global center-stable manifold at z' for the flow ϕ^t_X). Consider a small ball $B^{cs}(z, r) \subset V^{cs}_X(z)$ centered at z. By (3.11), the size of $\theta(B^{cs}(z, r))$ in the stable direction $E^s_{g^*}(z)$ of the disc is bounded by $C_1 r$ for some $C_1 > 3J$ independent of r. Since the stable foliation W^s_A of the torus is smooth, the size of $\theta(B^{cs}(z, r))$ in the stable direction $E^s_A(z)$ is bounded by $C_2 r$ for some $C_2 > 0$. Also, the size of $\theta(B^{cs}(z, r))$ in the central direction is bounded by $C_3 r$ for some $C_3 > 0$. Hence, the point z' cannot belong to $D(N \setminus \mathcal{A})$. Note that $z \notin \mathcal{E}$ implies $z' \in D(\mathcal{A}) \cup D(N \setminus \mathcal{A})$ and hence, $z' \in D(\mathcal{A})$.

We shall now show that $\mu(\mathcal{A}) = 1$. By Lemma 8, for almost every point $z' \in N \setminus \mathcal{E}$ one can find a point $z \in D(A) \setminus \mathcal{E}$ such that z and z' are accessible through a path $z_0, \dots, z_\ell$ such that $z_i \notin \mathcal{E}$. Repeating the above arguments we obtain that $z_1, \dots, z_\ell = z' \in D(\mathcal{A})$. Hence, $\mu(D(\mathcal{A})) = 1$. The desired result follows. $\square$

By identifying some boundary points, it is easy to see that the manifold N can be mapped onto the n-dimensional disc $\mathcal{D}^n$ via a map $\phi : N \to \mathcal{D}^n$ such that $\phi(N) = \mathcal{D}^n$ and $\phi|\mathrm{int}(N)$ is a diffeomorphism. Since $X|\partial N = 0$, $d\phi(X)$ is smooth on $\mathcal{D}^n$. There is also a mapping $\psi : \mathcal{D}^n \to M$ (see [K]), and the vector field $d\psi d\phi(X)$ generates the flow with the desired properties.

4. Proof of the theorem: the case $\dim M = 3$ and 4

In the case $\dim M = 3$, the proof is essentially the same. Consider the suspension flow over g with roof function 1. The suspension manifold $K = \mathcal{D}^2 \times [0, 1]/\sim$ (where $\sim$ is the identification $(x, 1) = (g(x), 0)$) is diffeomorphic to $N = \mathcal{D}^2 \times \mathcal{S}^1 = \{(x, t) : x \in M,\ t \in$

$[0,1]\}/\sim$ (where $\sim$ is the identification $(x,0)=(x,1)$). Let Z be the vector field of the suspension flow. For each $(x,t)\in K$ we have $Z=(0,1)$.

Let $F: K\to N$ be given by $F(x,t)=(G(x,t),t)$ (see Proposition 4). We have

$$dF(x,t)=\begin{pmatrix} G_x(x,t) & G_t(x,t) \\ 0 & 1 \end{pmatrix}.$$

Consider the vector field $Y=dFZ$. Note that

$$Y(G(x,t),t)=\Big(\frac{\partial G}{\partial t}(x,t),1\Big).$$

Define the vector field X on N by

$$X(G(x,t),t)=\left(\frac{\partial G}{\partial t}(x,t),\alpha(G(x,t))\right),$$

where $\alpha(x)$ is a C^∞ function on $\mathcal{D}^2$ satisfying Conditions $(A1)$ – $(A3)$. The vector field X is divergence-free and the flow ϕ_X^t has all the desired properties.

In the case $\dim M=4$ we start with a Bernoulli map with non-zero Lyapunov exponents on a 3-manifold constructed in [DP]. More precisely, define

$$S=g\times \mathrm{id}: \mathcal{D}^2\times \mathcal{S}^1\to \mathcal{D}^2\times \mathcal{S}^1.$$

Let $T(x,y)=(g(x),T_{\gamma(x)}y): \mathcal{D}^2\times\mathcal{S}^1\to\mathcal{D}^2\times\mathcal{S}^1$, where $T_{\gamma(x)}$ is rotation by $\gamma(x)$. Here $\gamma\colon \mathcal{D}^2\to\mathbb{R}$ is a nonnegative C^∞ function which is equal to zero in a small neighborhood of the discontinuity set $\overline{Q}=\{q_1,q_2,q_3,\partial\mathcal{D}^2\}\times\mathcal{S}^1$ and is positive elsewhere.

It is shown in [DP] that the function γ can be chosen so that the map T is *robustly accessible*, i.e., any C^1 perturbation R of T is accessible provided R coincides with T in a small neighborhood of the discontinuity set $\overline{Q}$. Moreover, there is a perturbation R of T which has non-zero Lyapunov exponents and is of the form $R=\phi\circ T$. Here the map ϕ differs from id in a small neighborhood of a point $z_0\in\mathcal{D}^2\times\mathcal{S}^1$ which lies outside a small neighborhood $\overline{U}$ of the set $\overline{Q}$. To describe the perturbation ϕ consider the coordinate system $\xi=\{\xi_1,\xi_2,\xi_3\}$ in an open disc $B(z_0,\epsilon)$ of asufficiently small radius ϵ centred at z_0 such that

1. $dm=d\xi$;
2. $E_T^c(z_0)=\partial/\partial\xi_1$, $E_T^s(z_0)=\partial/\partial\xi_2$, $E_T^u(z_0)=\partial/\partial\xi_3$.

Let $\psi(t)$ be a C^∞ function with support in $(-\epsilon,\epsilon)$. Set $\tau=\|\xi\|^2/\gamma^2$ and

$$\phi^{-1}(\xi)=(\xi_1\cos(\epsilon\psi(\tau))+\xi_2\sin(\epsilon\psi(\tau)),-\xi_1\sin(\epsilon\psi(\tau))+\xi_2\cos(\epsilon\psi(\tau)),\xi_3).$$

Choose a function $\widehat{\psi}(x,t):\mathbb{R}\times[0,1]\to\mathbb{R}$ with the following properties

1. $\widehat{\psi}(x,0)=0$ and $\widehat{\psi}(x,1)=\psi(x)$ for any $x\in\mathbb{R}$;
2. $\widehat{\psi}$ is C^∞ in x and t and $\widehat{\psi}'_t(x,t)=0$ for $t=0,1$ and $x\in\mathbb{R}$;
3. $\|\widehat{\psi}(x,t)\|_r<\delta$ for all $t\in[0,1]$.

Define the map $\Psi:\mathcal{D}^2\times\mathcal{S}^1\times[0,1]\to\mathcal{D}^2\times\mathcal{S}^1$ by the formula

$$\Psi(x,y,t)=\phi^t(x,y),$$

where

$$\phi^t(\xi)=(\xi_1\cos(\epsilon\widehat{\psi}(\tau,t))+\xi_2\sin(\epsilon\widehat{\psi}(\tau,t)),-\xi_1\sin(\epsilon\widehat{\psi}(\tau,t))+\xi_2\cos(\epsilon\widehat{\psi}(\tau,t)),\xi_3).$$

Note that Ψ has the properties similar to those of the map G in Proposition 4. Namely,

1. Ψ is C^∞ in (x,y,t);
2. $\Psi(\cdot,0)=\mathrm{id}$ and $\Psi(\cdot,1)=\phi$;

(3) for any $t \in [0,1]$ the map $\Psi(\cdot, t) : \mathcal{D}^2 \times \mathcal{S}^1 \to \mathcal{D}^2 \times \mathcal{S}^1$ is area-preserving;
(4) $\Psi_t(x, y, 1) = \Psi_t(\phi(x, y), 0)$;
(5) $\Psi(x, y, t) = \mathrm{id}$ for $x \in \overline{U}$.

Set
$$H = \{(x, y, t) : x \in \mathcal{D}^2, y \in \mathcal{S}^1, t \in [0,1]\}/\sim_1$$
with the identification $\sim_1$: $(x, y, 1) = (T(x, y), 0) = (g(x), T_{\gamma(x)}(y), 0)$ and
$$K = \{(x, y, t) : x \in \mathcal{D}^2, y \in \mathcal{S}^1, t \in [0,1]\}/\sim_2$$
with the identification $\sim_2$: $(x, y, 1) = (\phi \circ T(x, y), 0) = (\phi(g(x), T_{\gamma(x)}(y)), 0)$. Consider the diffeomorphism $\tilde{G} : H \to K$ given by
$$\tilde{G}(x, y, t) = (\Psi^{-1}(x, y, t), t),$$
where $\Psi^{-1}(x, y, t)$ for each t is the inverse of Ψ. Note that T itself is diffeotopic to S. Let $\bar{G}$ be the diffeomorphism $K \to K'$ where K' is the suspension manifold of the suspension flow over S. The manifold K' is diffeomorphic to $N = \mathcal{D}^2 \times S^1 \times S^1$. If $\widehat{G} : K' \to N$ then $F = \tilde{G} \circ \bar{G} \circ \widehat{G} : H \to N$ is a diffeomorphism.

Let Z be the vector field on H of the suspension flow over R . Obviously, $Z = (0, 0, 1)$ is divergence free. Set $Y = dFZ$. Since $F(x, y, t) = (G(x, t), y, t)$ for (x, y, t) in a neighborhood $\tilde{U}$ of the boundary of H then for $(x, y, t) \in \tilde{U}$,
$$Y(F(x, y, t)) = Y(G(x, t), y, t) = \left(\frac{\partial G}{\partial t}(x, t), 0, 1\right).$$
Let X be the vector field on N defined by the formula
$$X = \left(\frac{\partial G}{\partial t}(x, t), 0, \alpha(x)\right),$$
where $\alpha(x)$ satisfies Conditions $(A1)$ – $(A3)$. Clearly, X is divergence free and the flow generated by X has all the desired properties.

References

[BP] L. Barreira, Ya. Pesin *Lyapunov exponents and smooth ergodic theory*, University Lecture Series, **23**, AMS, 2002.

[B1] M. Brin, *Hölder continuity of invariant distributions*, in *Smooth ergodic theory and its applications*, 91–93, A. Katok, R. de la Llave, Ya. Pesin and H. Weiss eds., Proc. Symp. Pure Math., Amer. Math. Soc., 2001.

[B2] M. Brin, *Bernoulli diffeomorphisms with nonzero exponents*, Ergod. Th. & Dyn. Syst., **1**, (1981) 1–7.

[DHP] D. Dolgopyat, H. Hu, and Ya. Pesin, *An example of a smooth hyperbolic measure with countably many ergodic components*, in *Smooth ergodic theory and its applications*, 95–106, A. Katok, R. de la Llave, Ya. Pesin and H. Weiss eds., Proc. Symp. Pure Math., Amer. Math. Soc., 2001.

[DP] D. Dolgopyat and Ya. Pesin, *Every compact manifold carries a completely hyperbolic diffeomorphism* Ergod. Th. & Dyn. Syst., **22**, (2002), 409–435.

[K] A. Katok, *Bernoulli diffeomorphisms on surfaces*, Annals of Mathematics, **110**, (1979), 529–547.

[KB] A. Katok and K. Burns, *Infinitesimal Lyapunov functions, invariant cone families and stochastic properties of smooth dynamical systems*, Ergod. Th. & Dyn. Syst., **14**, (1994), 757–785.

[KH] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, Cambridge, 1995.

[M] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120**, (1965), 286–294.

[P1] Ya. Pesin *Characteristic Ljapunov exponents, and smooth ergodic theory*, Russian Math. Surveys **32**, (1977), 55–114.

[P2] Ya. Pesin, *A description of the* π*-partition of a diffeomorphism with an invariant measure. (Russian)*, Mat. Zametki **22**, (1977), 29–44.
[P3] Ya. Pesin, *Geodesic flows on closed Riemannian manifolds without focal points*, Math. USSR-Izv. **40**, (1977), 1195–1228.
[PS1] C. Pugh and M. Shub, *Stably ergodic dynamical systems and partial hyperbolicity*, J. of Complexity **13**, (1997), 125–179.
[PS2] C. Pugh and M. Shub, Pugh *Ergodic attractors*, Trans. Amer. Math. Soc. **312**, (1989), 1–54.
[S] S. Smale, *Diffeomorphisms of the* 2*-sphere*, Proc. Amer. Math. Soc., **10**, (1959), 621–626.

H. Hu, Department of Mathematics,Pennsylvania State University, University Park, PA 16802, USA
E-mail address: `hu@math.psu.edu`
Current address: Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
E-mail address: `hu@math.msu.edu`

Ya. Pesin, Department of Mathematics,Pennsylvania State University, University Park, PA 16802, USA
E-mail address: `pesin@math.psu.edu`

A. Talitskaya, Department of Mathematics,Pennsylvania State University, University Park, PA 16802, USA
E-mail address: `anjuta@math.psu.edu`