EQUILIBRIUMS OF SOME NON-HÖLDER POTENTIALS

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Abstract. We consider one-sided subshifts σ with some potential functions φ which satisfy the Hölder condition everywhere except at a fixed point and its preimages. We prove that the systems have conformal measures ν and invariant measures μ absolutely continuous with respect to ν, where μ may be finite or infinite. We show that the systems (σ, μ) are exact, and μ are weak Gibbs measures and equilibriums for φ. We also discuss uniqueness of equilibriums and phase transition.

These results can be applied to some expanding dynamical systems with an indifferent fixed point.

0. Introduction

The motivation of the paper is to understand statistical properties of physical measures for almost expanding dynamical systems with Markov partitions. We say that a piecewise smooth system is *almost expanding* if it is expanding everywhere except at a finite number of periodic orbits. Examples of such systems are given in Section 2, which include piecewise expanding maps on the unit interval, parabolic rational maps on Julia sets, etc. We only consider the case where the systems contain one indifferent fixed point p. Systems with more indifferent fixed points or periodic orbits can be treated similarly.

Since we assume that the systems have Markov partitions, they can be represented by a one-sided subshift of finite type, and we can work on potentials φ. With the usual metric on symbolic space, the potentials we study do not satisfy Hölder conditions at the fixed point and its preimages. Therefore, statistical properties of the systems become different from those with Hölder potentials.

We obtain existence of a conformal measure ν and an invariant measure μ for such a potential, where μ is a physical measure of the system we are interested in, and study the properties of the measures. We show that such a system (σ, μ) is exact, and therefore is ergodic if the symbolic system is topologically mixing. We prove that μ is a weak Gibbs measure, and obtain conditions under which the formula $P(\sigma, \varphi) = h_\mu(\sigma) + \mu(\varphi)$ holds. We also study uniqueness of equilibriums

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and phase transition. Lastly, we give the rates of convergence, without proof, of test functions to their equilibriums under the transfer operators.

There are many literatures related to the topic. For non-Hölder potentials, existence and uniqueness of equilibriums, and rates of convergence to the equilibriums are studied by Hofbauer, Fisher-Lopes, Maume-Deschamps and others (see e.g. [Ho], [M], [KMS], [N], [FL]). In those references, potentials are usually assumed to be piecewise constant, or to be summable. The systems with indifferent fixed points are sometimes coded by a subshift with countably many states (see e.g. [S1]-[S4] and their references). Also, some ergodic properties have been studied for the systems discussed in Section 2. For piecewise expanding maps with an indifferent fixed point on the unit interval, existence and condition for finiteness of absolutely continuous invariant measures was proved by Pianigiani ([Pi]) and Thaler ([T]). When the invariant measure is finite, weak Gibbsianness, thermodynamic formalism, and phase transition have been studied by M. Yuri (see e.g. [Yu1]-[Yu4] and their references). When it is infinite, ergodic properties were studied by Zweimüller ([Z]). Parabolic rational maps on Julia sets have been studied by Denker-Urbański (see e.g. [DU2], [DU3]). These results were extended to parabolic Cantor sets by Urbański (see [U1] and its references). Rates of convergence to equilibriums and rates of decay of correlations of systems were studied in [Y2], [H1], [S4], [G], and others (see [H2] for more references).

In this paper, we try to give the simplest conditions on the potential functions and to obtain most ergodic properties for varieties of almost expanding systems. Hence, among the systems discussed in Section 2, an ergodic property found in one kind of system may also hold for others. The conditions we give are weaker since for most results we do not need Assumption(III$'$), i.e. we only need the lower bound of $|\varphi(\bar{0}) - \varphi(x)|$. Further, the potentials we study are more general. For example, in one dimensional almost expanding systems, our potentials are not necessary to have the form $-t \log f'(x)$. We may have bounded density functions and exponential rates of convergence to the equilibriums. In this case, the behaviors of the systems are just like those with Hölder potentials. We may also have unbounded density functions, finite or infinite invariant measure. In this case, the measures of the tail of the Young's tower may or may not converge to 0 (see Corollary A.2), and the rates of convergence to the equilibriums are only polynomial. Moreover, we prove exactness and study Gibbs property in both cases where μ are finite and infinite. We prove uniqueness of weak Gibbs state and equilibrium, and give a complete description for phase transition. Our main approach avoids the first return maps and goes down to Bowen's method.

This paper is organised as follows. The assumptions and results are stated in Section 1. In Section 2 we apply these results to piecewise smooth almost expanding maps. In Section 3 we prove Theorem A, that mainly deals with existence of conformal measures and invariant measures. In Section 4 we discuss properties of the density functions of the invariant measures, which are stated in Corollary A.1. The measures of the tail of tower are estimated in Section 5, Section 6 is for exactness and a proof of Theorem B, while Section 7 is for Gibbs properties and a proof of Theorem C. The last section, deals with equilibriums and uniqueness.

1. Assumptions, statements of results and notations

Let $\Sigma^+ = \prod_0^\infty \{0, 1, \cdots, r^* - 1\}$ and $\sigma : \Sigma^+ \to \Sigma^+$ be the left shift.

If A is an $r^* \times r^*$ matrix of 0's and 1's, let $\Sigma_A^+ = \{x \in \Sigma^+ : \ A_{x_i x_{i+1}} = 1 \ \forall i \ge 0\}$. It is well known that $\sigma\Sigma_A^+ = \Sigma_A^+$. We assume that σ is topologically mixing. We also assume $A_{00} = 1$, so, $\bar{0} = 000\cdots$ is a fixed point of σ.

For convenience we assume that $A_{11} = 1$ so that $\bar{1} = 111\cdots$ is another fixed point. We can check that the results are still true without the assumption.

We say that w is an n-word if $w = w_0 w_1 \cdots w_{n-1}$ and $A_{w_i w_{i+1}} = 1 \ \forall 0 \le i < n-1$. The word uw is the word u followed by the word w.

Given an n-word $w = w_0 w_1 \cdots w_{n-1}$, we define

$$\mathcal{R}_w = \{x \in \Sigma_A^+ : \ x_i = w_i, \ \forall 0 \le i \le n-1\}.$$

This set is called an n-cylinder, or simply a cylinder.

Let ξ be the partition of Σ_A^+ into $\{\mathcal{R}_s : s = 0, 1 \cdots r^* - 1\}$, and

$$\xi_n = \bigvee_{i=0}^{n-1} \sigma^{-i}\xi. \tag{1.1}$$

For any n-word w, we have $\mathcal{R}_w \in \xi_n$. We simply write $w \in \xi_n$ instead.

For $k \ge 0$, we denote $\mathcal{O}_k = \mathcal{R}_{0^k}$, $P_k = \mathcal{O}_k \backslash \mathcal{O}_{k+1}$, and $Q_k = \Sigma_A^+ \backslash \mathcal{O}_{k+1}$. In other words, $\mathcal{O}_k$, P_k and Q_k are sets of the points that start with at least, exact, and at most k zeros, respectively. Also, we denote $P'_k = P_{k-1} \cup P_k \cup P_{k+1}$.

Take $\kappa \in (0, 1)$ and $\gamma > 0$. Let K_0 be the largest number k such that $\kappa^k \ge (k+1)^{-(1+\gamma)}$. Define a metric on Σ_A^+ inductively by the following rules:

i) $d(x, y) = 1$ if $x_0 \ne y_0$.
ii) $d(x, y) = \kappa d(\sigma x, \sigma y)$ if $x_0 = y_0$ and $x, y \in Q_{K_0}$.
iii) $d(x, y) = (k+1)^{-(\gamma+1)} d(\sigma^k x, \sigma^k y)$ if $x \in P_k$, $y \in P'_k$, $k > K_0$.
iv) $d(x, y) = \sum_{i=k}^{k+l-1} d(x^{(i)}, x^{(i+1)})$, if $x \in P_k$ and $y \in P_{k+l}$, $k \ge K_0$, where $x^{(k)} = x$, $x^{(k+l)} = y$, and $x^{(i)} \in P_i$ for $i = k+1, \cdots, k+l-1$.

With this metric, the left shift $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is uniformly expanding with a rate κ^{-1} on Q_{K_0}. The expanding rate of σ on P_k converges to 1 if $k \to \infty$.

By the metric we can see that if $x = 0^k\tilde{x} \in P_k$, $y = 0^k\tilde{y} \in P'_k$ such that $\tilde{x}_0 \ne \tilde{y}_0$, then $d(x, y) = \max\{\kappa^k, (k+1)^{-(\alpha+1)}\}$. Hence, there exists $C_\gamma \ge 1$ such that

$$\operatorname{diam} \mathcal{O}_k \le C_\gamma k^{-\gamma} \qquad \forall k \ge 0, \tag{1.2}$$

where the diameter of a set S is defined by $\operatorname{diam} S = \sup\{d(x, y) : x, y \in S\}$.

We assume that the potential function φ satisfies the following.

Assumption A.

(I) *φ is a continuous function on Σ_A^+;*
(II) *$\exists\, \theta \in (0, 1]$, $\alpha \in [0, \theta(1+\gamma))$ and $C_\varphi > 0$ such that*

$$|\varphi(x) - \varphi(y)| \le C_\varphi \max\{K_0^{\alpha-1}, k^{\alpha-1}\} d(x, y)^\theta \qquad \forall x \in P_k, y \in P'_k;$$

(III) $\exists\ \beta > -1,\ K_1 > 0$ *such that for all* $k \ge K_1$,

$$\varphi(\bar{0}) - \varphi(x) \ge \frac{\beta+1}{k+1} - \frac{C_\delta}{(k+1)^{1+\delta}} \qquad \forall x \in P_k$$

for some constants $\delta > 0$ *and* $C_\delta > 0$ *independent of* k *and* x.

Sometimes we also assume

(III′) $\exists\ \beta' \ge \beta,\ K_1' > 0$ *such that for all* $k \ge K_1'$,

$$\varphi(\bar{0}) - \varphi(x) \le \frac{\beta'+1}{k+1} + \frac{C_\delta}{(k+1)^{1+\delta}} \qquad \forall x \in P_k.$$

We may assume $\delta \le \min\{1, \gamma\theta\}$ since we can always reduce δ.

We will also assume $\varphi(\bar{0}) = 0$ since otherwise we can use $\varphi(x) - \varphi(\bar{0})$ instead.

Remark 1.1. With the standard metric $\tilde{d}(x,y) = 2^{-k}$, where $k = \min\{i : x_i \ne y_i\}$ for $x = \{x_i\}$ and $y = \{y_i\}$, φ is not a Hölder function because of Assumption A(III). However, under the metric we define, φ satisfies the Hölder condition.

Remark 1.2. If we denote

$$\operatorname{var}_k(\varphi) = \max\{|\varphi(x) - \varphi(y)| :\ x_i = y_i, \forall i = 0, \cdots, k-1\}, \tag{1.3}$$

then by Assumption A(III) $\operatorname{var}_k(\varphi) \ge Ck^{-1}$ for some $C > 0$. So the potential φ does not have summable variations since $\sum\limits_{k=0}^{\infty} \operatorname{var}_k(\varphi)$ is not summable.

Let $C^0(S)$ denote the set of continuous real functions on the set S. Define the Perron-Frobenius Operator $\mathcal{L}_\varphi$ from $C^0\big(\Sigma_A^+ \backslash \{\bar{0}\}\big)$ or $C^0\big(\Sigma_A^+\big)$ to itself by

$$\mathcal{L}_\varphi g(x) = \sum_{y \in \sigma^{-1}x} e^{\varphi(y)} g(y).$$

Denote by $\mathcal{L}_\varphi^*$ the dual operator of $\mathcal{L}_\varphi$ on $\mathcal{M}(\Sigma_A^+)$.

Denote by $\tilde{\varphi}$ the first return map with respect to P_0, and by $\tilde{\varphi}$ the corresponding potential, that is, for any $x \in P_0$, $\tilde{\sigma}x = \sigma^n x$ and $\tilde{\varphi}(x) = S_n\varphi(x)$ where $n = n(x)$ is the smallest positive integer such that $\sigma^n x \in P_0$. The corresponding Perron-Frobenius Operator $\tilde{\mathcal{L}}_{\tilde{\varphi}}$ is given by

$$\tilde{\mathcal{L}}_{\tilde{\varphi}} g(x) = \sum_{\tilde{\sigma}y = x} e^{\tilde{\varphi}(y)} g(y) = \sum_{j=1}^{\infty} \sum_{s \ne 0} e^{S_j\varphi(s0^{j-1}x)} g(s0^{j-1}x). \tag{1.4}$$

Denote $D_k = \max\{k, K_0\}$. For $J \ge 0$, we define

$$\overline{\mathcal{G}}_J = \{g \in C^0\big(\Sigma_A^+ \backslash \{\bar{0}\}\big) : g > 0,\ g(y) \le g(x) e^{JD_k^\alpha d(x,y)^\theta}\ \forall x \in P_k, y \in P_k', \forall k \ge 0\}.$$

Denote by $\mathcal{M}(\Sigma_A^+)$ the set of Borel probability measures on Σ_A^+.

Theorem A (Existence of the invariant measures). *Suppose φ satisfies either Assumption A(I)-(III) with $\beta > 0$, or Assumptions A(I)-(III) and (III$'$) with $-1 < \beta \le \beta' \le 0$. Then there is a measure $\nu \in \mathcal{M}(\Sigma_A^+)$, which is positive on nonempty open sets, a constant $\lambda \ge e^{\varphi(\bar{0})} = 1$, $\lambda > 1$ if $-1 < \beta' \le 0$, and a function $h \in \overline{\mathcal{G}}_{J_\varphi}$ for some $J_\varphi > 0$ such that $\mathcal{L}_\varphi^* \nu = \lambda \nu$, $\mathcal{L}_\varphi h = \lambda h$.*

Moreover, $\mu(g) = \nu(hg)$ defines a finite or infinite σ-invariant measure μ.

- *μ is finite if either $h^*(\bar{1}) < 0$, or $h^*(\bar{1}) = 0$ and $\beta > 1$;*
- *μ is infinite if either $h^*(\bar{1}) > 0$, or $h^*(\bar{1}) = 0$ and Assumption A(III$'$) holds with $0 < \beta' \le 1$,*

where

$$h^*(x) = h(x) - \sum_{j=1}^{\infty} \sum_{s \ne 0} e^{S_j \varphi(s0^{j-1}x)} h(s0^{j-1}x). \tag{1.5}$$

If μ is finite, then we assume that μ is a probability measure.

By Corollary C.1, we see that $\log \lambda$ is the topological pressure $P(\sigma, \varphi)$ for the potential function φ. We will prove in Lemma 4.3 that the sign of h^* is independent of x, and $\lambda > 1$ if and only if $h^* < 0$.

Remark 1.3. Let $\tilde{\sigma}$ be the first return map and $\tilde{\varphi}(x)$ the corresponding potential. If the topological pressure $P(\tilde{\sigma}, \tilde{\varphi})$ can be defined, then one should expect that $P(\tilde{\sigma}, \tilde{\varphi})$ and h^* have the opposite sign.

Remark 1.4. Sometimes the measure ν is called an $e^{\log \lambda - \varphi}$-*conformal measure* in the sense that for any Borel set E such that $\sigma|_E$ is injective,

$$\nu(\sigma E) = \int_E \epsilon^{\log \lambda - \varphi} d\nu. \tag{1.6}$$

For any function g defined on Σ_A^+ or $\Sigma_A^+ \backslash \{\bar{0}\}$, we denote

$$\hat{g}(0x) = \sum_{s \ne 0} e^{\varphi(sx)} g(sx). \tag{1.7}$$

Corollary A.1 (Properties of the density function).

i) *If $h^*(\bar{1}) < 0$, then $\lim_{x \to \bar{0}} h(x) = \dfrac{\hat{h}(\bar{0})}{\lambda - 1}$; otherwise $\lim_{x \to \bar{0}} h(x) = \infty$.*

ii) *If $h^*(\bar{1}) = 0$, then for any $x \ne \bar{0}$, $\limsup_{n \to \infty} \dfrac{\beta}{n} h(0^n x) \le \hat{h}(\bar{0})$. Suppose Assumption A(III$'$) also holds; then $\liminf_{n \to \infty} \dfrac{\beta'}{n} h(0^n x) \ge \hat{h}(\bar{0})$. So if $\beta = \beta'$, then the* lim sup *and* lim inf *become limit.*

iii) *If $h^*(\bar{1}) > 0$, then there exists $B_h > 0$ such that $\liminf_{n \to \infty} \dfrac{1}{n^{\beta+1}} h(0^n x) \ge B_h h^*(x)$ for any $x \in P_0$. Suppose Assumption A(III$'$) also holds; then there exists $B_h' > 0$ such that $\limsup_{n \to \infty} \dfrac{1}{n^{\beta'+1}} h(0^n x) \le B_h' h^*(x)$ for any $x \in P_0$. Further, if $\beta = \beta'$, then $\lim_{n \to \infty} \dfrac{1}{n^{\beta+1}} h(0^n x) = B_h^* h^*(x)$ for some $B_h^* > 0$.*

Since functions in $\overline{\mathcal{G}}_{J_\varphi}$ are undefined at $\bar{0}$, by Corollary A.1 and Lemma 4.1, we can define

$$h(\bar{0}) = \hat{h}(\bar{0})(\lambda - 1)^{-1} \ \text{ if } \lambda > 1, \quad \text{and} \ \ h(0) = \infty \ \text{ otherwise} \tag{1.8}$$

It is well known that the convergence rates of the tail of Young's tower ([Y1], [Y2]) determine the rates of convergence of test functions to the equilibrium, and the rates of decay of correlations.

Corollary A.2 (Convergence rates of the tail)**.**

For any $k \ge 0$, there exist $B_\mu = B_{\mu,k}$, $B'_\mu = B'_{\mu,k}$, $C_\mu = C_{\mu,k} > 0$ and $C'_\mu = C'_{\mu,k} > 0$ such that the limits $\lim\limits_{k\to\infty} A_{\mu,k}$, where $A = B, B', C, C'$ exist, and for all $n \ge k$:

i) *if $h^*(\bar{1}) < 0$, then $\mu P_n \le B_{\mu,k}\lambda^{-n} n^{-(\beta+1)}$ and $\mu\mathcal{O}_n \le C_{\mu,k}\lambda^{-n}n^{-\beta}$;*
ii) *if $h^*(\bar{1}) = 0$, then $\mu P_n \le B_{\mu,k}n^{-\beta}$ $\forall \beta > 0$, $\mu\mathcal{O}_n \le C_{\mu,k}n^{-(\beta-1)}$ $\forall\beta > 1$;*
iii) *if $h^*(\bar{1}) > 0$, then $\lim\limits_{n\to\infty} \mu P_n > 0$ and $\mu\mathcal{O} = \infty$.*

Suppose Assumption A (III′) also holds; then i) and ii) are true if we replace $B_{\mu,k}$, $C_{\mu,k}$ and "$\le$" by $B'_{\mu,k}$, $C'_{\mu,k}$ and "$\ge$" respectively, and for case ii) it also holds that $\mu\mathcal{O}_n = \infty$ $\forall 0 < \beta' \le 1$.

Moreover, if $\beta = \beta'$, then $B_{\mu,k}$ and $B'_{\mu,k}$, and $C_{\mu,k}$ and $C'_{\mu,k}$ can be chosen in such a way that the following limits exist: $\lim\limits_{k\to\infty} B_{\mu,k} = \lim\limits_{k\to\infty} B'_{\mu,k} = \lim\limits_{n\to\infty} \lambda^n n^{\tilde\beta}\nu P_n$ for $\tilde\beta > 0$, and $\lim\limits_{k\to\infty} C_{\mu,k} = \lim\limits_{k\to\infty} C'_{\mu,k} = \lim\limits_{n\to\infty} \lambda^n n^{\tilde\beta-1}\nu\mathcal{O}_n$ for $\tilde\beta > 1$, where $\tilde\beta = \beta + 1$ in case i) and $\tilde\beta = \beta$ in case ii).

Estimates for νP_n and $\nu\mathcal{O}_n$ are given in Lemma 4.5.

Define $\psi(x) = \varphi(x) + \log h(x) - \log h(\sigma x) - \log\lambda$ for $x \in \Sigma_A^+$, where we regard $\log h(\bar{0}) - \log h(\sigma\bar{0}) = 0$ if $h(\bar{0}) = \infty$.

It is easy to see that $\psi(x) = -\log \dfrac{d\mu\circ\sigma}{d\mu}(x)$, while $\varphi(x) = -\log\dfrac{d\nu\circ\sigma}{d\nu}(x)$. So we can put assumption on ψ instead of φ to get statistic properties of the systems (see [H2]). We only state the results for the case $h^*(\bar{1}) = 0$, since this is the most interesting case.

Note that if $x \in \sigma^{-1}(\sigma P_k)$, then x has the form $s0^{k-1}w$, where $w = w_0w_1\cdots$ with $w_0 \ne 0$.

Corollary A.3 (Properties of the function ψ)**.**

Consider the case $h^*(\bar{1}) = 0$.

(I) *ψ is continuous on each $\mathcal{R}_s$ except at $s\bar{0}$, $s \ne 0$.*
(II) *$\exists\ J_\psi > 0$ such that for all n-words $w = w_0w_1\cdots w_{n-1}$,*

$$\big|S_n\psi(wx) - S_n\psi(wy)\big| \le J_\psi \max\{K_0^\alpha, k^\alpha\} d(x,y)^\theta \qquad \forall x \in P_k, y \in P'_k,$$

and if $w_{n-1} \ne 0$, then

$$\big|\big(S_n\psi(wx) + \log h(x)\big) - \big(S_n\psi(wy) + \log h(y)\big)\big| \le J_\psi K_0^\alpha d(x,y)^\theta \quad \forall x, y \in \Sigma_A^+.$$

(III) $\exists\, K_2 > 0$ *such that for all* $k \geq K_2$,

$$\psi(\bar{0}) - \psi(x) \geq \frac{\beta}{k} - \frac{C_\delta}{k^{1+\delta}} \qquad \forall x \in P_k$$

for some $C_\delta > 0$ *independent of* k *and* x.

Moreover, if Assumption A (III′) also holds and $\lambda = 1$, *then*

(III′) $\exists\, K_2' > 0$ *such that for all* $k \geq K_2'$,

$$\psi(\bar{0}) - \psi(x) \leq \frac{\beta'}{k} + \frac{C_\delta}{k^{1+\delta}} \qquad \forall x \in P_k$$

for some $C_\delta' > 0$ *independent of* k *and* x.

Recall that a σ-invariant measure μ is *ergodic* if for any measurable set E, $\sigma^{-1}E = E \pmod{\mu}$ implies $\mu E = 0$ or $\mu(\Sigma_A^+ \setminus E) = 0$. A system (σ, μ) is *exact* if the tail σ-algebra $\bigcap_{n=1}^{\infty} \sigma^{-n}\mathcal{B}$ is trivial, where $\mathcal{B}$ is the σ-algebra for the system (σ, μ). These definitions work for both probability and infinite measures. (See [A] for the infinite measure case.) It is well known that exactness implies ergodicity. Also, if μ is a probability measure, then exactness implies mixing.

Theorem B (Ergodicity and exactness)**.** *Under the assumptions of Theorem A, the system* (σ, μ) *is exact. Therefore,* μ *is an ergodic measure.*

Recall that a measure ρ is a *Gibbs measure*, if there exist constants P and C such that for any x, and $n \geq 0$,

$$C^{-1} \leq \frac{\rho \mathcal{R}_{x_0 x_1 \cdots x_{n-1}}}{\exp\{-nP + S_n\varphi(x)\}} \leq C.$$

In our case one cannot expect that the measure μ obtained in Theorem A is a Gibbs measure, since if we take $x = \bar{0}$, then by Corollary A.2, $\mu\mathcal{O}_n$ may decrease polynomially, while $\exp\{-nP + n\varphi(\bar{0})\} = e^{-nP}$ decreases exponentially if $P > 0$ or equal to 1 if $P = 0$. However, μ is a week Gibbs measure. A measure ρ is a *weak Gibbs measure*, if there exists a constant P and a sequence $\{C_n\}$ with $\lim_{n\to\infty} \frac{1}{n} \log C_n = 0$ such that for any x, and $n \geq 0$,

$$C_n^{-1} \leq \frac{\rho \mathcal{R}_{x_0 x_1 \cdots x_{n-1}}}{\exp\{-nP + S_n\varphi(x)\}} \leq C_n. \tag{1.9}$$

We refer to [Yu2] and [Yu3] and their references for more information about weak Gibbs measures.

We can extend the definition for σ-finite measures. An invariant measure ρ is said to be an *infinite weak Gibbs measure* if $\rho(\Sigma_A^+) = \infty$, and (1.9) holds for all $x \in \Sigma_A^+$ and $n > 0$ provided $\rho\mathcal{R}_{x_0 x_1 \cdots x_{n-1}} < \infty$.

Theorem C (Gibbs properties)**.** *Under the assumptions of Theorem A, μ is a weak Gibbs measure or an infinite weak Gibbs measure. Moreover, μ is the unique invariant measure satisfies the following properties: There is a real number P, and a function $p(x,n)$ such that for every x,*

$$\frac{1}{p(x,n)} \le \frac{\mu\mathcal{R}_{x_0x_1\cdots x_{n-1}}}{\exp\{-nP+S_n\varphi(x)\}} \le p(x,n), \tag{1.10}$$

provided $\mu\mathcal{R}_{x_0x_1\cdots x_{n-1}} < \infty$, where $p(x,n)$ satisfies the following:

a) $\lim_{n\to\infty} \frac{1}{n}\log p(x,n) = 0$ *for any* $x \ne \bar{0}$;

b) $\lim_{k\to\infty} \frac{1}{k}\log \sup_{x\in Q_k\cap\sigma^{-n+1}Q_0} p(x,n) = 0.$

Remark 1.5. Part b) implies that for each k, we can find uniform bounds for cylinders $\mathcal{R}_w$ if $\mathcal{R}_u \subset Q_k$ and the last symbol of w is nonzero. If we think that such cylinders are "good" cylinders, then for any $x \in Q_k$ which is not a preimage of $\bar{0}$, $\mathcal{R}_{x_0x_1\cdots x_{n-1}}$ is a "good" cylinder for infinitely many n.

Corollary C.1 (The constant P)**.** *The constant P in both (1.9) and (1.10) is equal to the topological pressure $P(\sigma,\varphi)$ and $\log\lambda$, where λ is given in Theorem A.*

For entropy of a σ-finite measure ρ, we follow the definition given by Krengel (see [Kr], also [Z]). For a subset $\Gamma \subset \Sigma_A^+$, we denote by σ_Γ the corresponding first return map, and by ρ_Γ the conditional measure of ρ, that is, $\rho_\Gamma S = \rho S/\rho\Gamma$ for $S \subset \Gamma$. The *measure theoretic entropy* of ρ is defined by

$$h_\rho(\sigma) = \rho(\Gamma)h_{\rho_\Gamma}(\sigma_\Gamma) \tag{1.11}$$

for any subset Γ of positive finite measure.

Recall that a probability measure ρ is an *equilibrium state* for a potential η if it satisfies

$$P(\sigma,\eta) = h_\rho(\sigma) + \int \eta d\rho. \tag{1.12}$$

We also denote $P(\eta) = P(\sigma,\eta)$.

Theorem D (Equilibrium states)**.** *Under the assumptions of Theorem A,*

- *μ satisfies (1.12) with $\eta = \varphi$ if and only if $h^*(\bar{1}) \le 0$;*
- *the Dirac measure $\delta_{\bar{0}}$ satisfies (1.12) with $\eta = \varphi$ if and only if $h^*(\bar{1}) \ge 0$.*

Further, the only ergodic (probability) equilibrium for φ is

- *μ if $P(\varphi) > 0$;*
- *μ and $\delta_{\bar{0}}$ if $P(\varphi) = 0$ and $\mu\Sigma_A^+ = 1$;*
- *$\delta_{\bar{0}}$ if $P(\varphi) = 0$ and $\mu\Sigma_A^+ = \infty$.*

Suppose Assumption A (III$'$) also holds; then μ is the only possible infinite ergodic measure ρ with $|\rho(\varphi)| < \infty$ such that (1.12) holds with $\eta = \varphi$.

By Theorem A and Corollary C.1, if μ is an infinite measure, then $P(\sigma,\varphi) = 0$. So Theorem D provides conditions for the Rohlin's formula $h_\mu(\sigma) = -\int \varphi d\mu$.

Corollary D.1 (Phase transition). *Suppose $\varphi \le 0$ satisfies Assumptions A (I)-(III) and (III$'$) with $\beta = \beta'$. Then there exists $t_0 > 0$ such that*

i) *for $0 \le t < t_0$, $P(t\varphi) > 0$ and φ has the unique equilibrium $\mu_{t\varphi}$;*
ii) *for $t = t_0$, $P(t\varphi) = 0$ and φ has exactly two equilibriums $\mu_{t\varphi}$ and $\delta_{\bar{0}}$ if $t_0(\beta+1) > 2$, and has the unique equilibrium $\delta_{\bar{0}}$ otherwise;*
iii) *for $t > t_0$, $P(t\varphi) = 0$ and φ has the unique equilibrium $\delta_{\bar{0}}$.*

Corollary D.2 (Uniqueness of weak Gibbs measures). *Under the assumptions of Theorem A, if μ is a probability measure, then it is the only σ-invariant weak Gibbs measure for φ.*

We define $\mathcal{L}_\psi : C^0(\Sigma_A^+) \to C^0(\Sigma_A^+)$ by $\mathcal{L}_\psi g = \dfrac{1}{\lambda h}\mathcal{L}_\varphi(hg)$, or, equivalently

$$\mathcal{L}_\psi g(x) = \sum_{y\in\sigma^{-1}x} e^{\psi(y)} g(y) = \frac{1}{\lambda h(x)} \sum_{y\in\sigma^{-1}x} e^{\varphi(y)} h(y) g(y).$$

If $h^*(\bar{1}) \le 0$, the convergence rate of a test function to its equilibrium under the operators $\mathcal{L}_\psi$ is determined by $P(\varphi)$ and β. We state the results here.

Denote

$$\begin{aligned}
\mathcal{G} =& \big\{g \in C^0\big(\Sigma_A^+\big) : \ \exists C > 0, \text{ s.t. } |g(y) - g(x)| \le C D_k^\alpha d(x,y)^\theta \\
& \qquad\qquad\qquad\qquad\qquad \forall x \in P_k, y \in P_k' \ \forall k \ge 0\big\},\\
\mathcal{G}_0 =& \big\{g \in \mathcal{G} : \ g(\bar{0}) \ne \mu(g)\big\},\\
\mathcal{G}_\delta =& \big\{g \in \mathcal{G} : \ \exists L > 0, \text{ s.t. } |g(\bar{0}) - \mu(g)| \le L(n+1)^{-\delta} \ \forall n \ge 0\big\}.
\end{aligned}$$

Let $\mathcal{F}(\Sigma_A^+)$ be the set of all bounded real functions on Σ_A^+. Denote

$$\begin{aligned}
\mathcal{F}_\tau^+ =& \left\{g \in \mathcal{F}(\Sigma_A^+)) : \ \exists L > 0, \text{ s.t. } |g(x) - g(\bar{0})| \le L(n+1)^{-\tau} \ \forall x \in P_n, n \ge 0\right\},\\
\mathcal{F}_\tau^- =& \left\{g \in \mathcal{F}(\Sigma_A^+)) : \ \exists L' > 0, \text{ s.t. } g(x) - g(\bar{0}) \ge L'(n+1)^{-\tau} \ \forall x \in P_n, n \ge 0\right\}.
\end{aligned}$$

Theorem E (Rates of convergence). *Suppose φ satisfies Assumptions A(I)-(III).*

If $h^*(\bar{1}) < 0$, then there is $\tilde{\lambda} > 0$ such that for any $g \in \mathcal{G}$, there is $A > 0$ with

$$|\mathcal{L}_\psi^n g(x) - \mu(g)| \le A\tilde{\lambda}^{-n} \qquad \forall x \in Q_k, n \ge 0.$$

If $h^*(\bar{1}) = 0$ and $\beta > 1$, then for any $g \in \mathcal{G}_\tau$, $\tau \in [0,1]$, there is $A > 0$ such that

$$|\mathcal{L}_\psi^n g(x) - \mu(g)| \le \frac{A}{(n+1)^{\min\{\beta-1+\tau,\beta\}}} \qquad \forall x \in Q_k, n \ge 0.$$

Moreover, if Assumption A (III$'$) also holds with $\beta' \le \beta + 1$, then for any $g \in \mathcal{G}_0$, there is $A' > 0$ such that

$$|\mathcal{L}_\psi^n g(x) - \mu(g)| \ge \frac{A'}{(n+1)^{\beta'-1}} \qquad \forall x \in Q_k, n \ge 0.$$

The above inequalities are also true if we replace $|\mathcal{L}_\psi^n g(x) - \mu(g)|$ *by* $\|\mathcal{L}_\psi^n g - \mu(g)\|$.

If $h^*(\bar{1}) = 0$ *and Assumptions A (III′) also holds with* $\beta = \beta' \in (0, 1]$*, then for any* $g \in \mathcal{F}_\tau^+$*, there is* $A > 0$ *such that*

$$|\mathcal{L}_\psi^n g(x) - g(\bar{0})| \le \frac{A(n+1)^{\max\{0, 1-\beta-\tau\}}}{(n+1)^{1-\beta}} \qquad \forall x \in \Sigma_A^+,\ n \ge 0;$$

and for any $g \in \mathcal{F}_\tau^-$*, there is* $A' > 0$ *such that*

$$\mathcal{L}_\psi^n g(x) - g(\bar{0}) \ge \frac{A'(n+1)^{\max\{0, 1-\beta-\tau\}}}{(n+1)^{1-\beta}} \qquad \forall x \in Q_n,\ n \ge 1,$$

where $(n+1)^{1-\beta}$ *or* $(n+1)^{1-\beta-\tau}$ *in the inequalities should be replaced by* $\log(n+1)$ *if* $\beta = 1$ *or* $\beta + \tau = 1$*, respectively.*

We are not going to prove the theorem in this paper. For the case $P(\varphi) > 0$, we have $\lambda > 1$. By Corollary A.2, μP_k and $\mu \mathcal{O}_n$ decreases exponentially fast. Then we can apply results of Young ([Y1]) to get exponential convergence. For the case $P(\varphi) = 0$, the results and more details can be seen in [H2] and [HH] for the case $\beta > 1$ and $\beta = \beta' \le 1$ respectively. (See also [Y2], [S4], [G] for the case $\beta > 1$.)

From this theorem, we can get corresponding results for rates of decay of correlations when the invariant measure is finite. Further, if the covariance

$$\mu(g \cdot g \circ \sigma^n) - \mu(g)^2$$

is summable with n, then the Central Limit Theorem holds. In our case, if $\beta - 1 + \tau > 1$, then the Central Limit Theorem holds for any $g \in \mathcal{G}_\tau$.

2. Almost expanding maps: Applications

Consider a map $f : X \to X$, where $X = \mathbb{R}^m$ or $\bar{\mathbb{C}}$, the Riemannian sphere. Suppose f has an invariant subset Λ, i.e. $f\Lambda = \Lambda$.

Assumption B.

(I) $f|_\Lambda : \Lambda \to \Lambda$ *is topologically mixing.*

(II) $f|_\Lambda : \Lambda \to \Lambda$ *has a Markov partition into subsets* $\{R_i\}_{i=0}^{r^*-1}$.

(III) f *is piecewise smooth. More precisely, for each* i, $f|_{\operatorname{int} R_i}$ *is a* C^2 *map from* $\operatorname{int} R_i$ *to its image, and it can be* C^1 *extended to* $\overline{R_i}$.

Here a Markov partition means a finite cover $\{R_0, \cdots, R_{r^*-1}\}$, $R_i = \overline{\operatorname{int}_\Lambda R_i}$, of Λ with $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset \ \ \forall i \ne j$; and $f(R_i)$ is a union of some R_j.

A fixed point p of f is *indifferent* if Df_p has an eigenvalue on the unit circle in $\mathbb{C}$. We say that the map f is expanding on an invariant set Λ with an indifferent fixed point p if f has an indifferent fixed point $p \in \Lambda$ and f is uniformly expanding away from p. The latter means that for any open neighbourhood U of p, f is uniformly expanding on $\Lambda \backslash U$.

If f is expanding on an invariant subset Λ, with or without indifferent fixed points, and has a Markov partition, then there is a map $\pi : \Sigma_A^+ \to \Lambda$ given by $\pi(\{x_i\}) = \bigcap_{j=0}^{\infty} f^{-i} R_{x_j}$, where $\{x_i\} \in \Sigma_A^+$, such that $f \circ \pi = \pi \circ \sigma$.

Assume that f has an indifferent fixed point p. Then we can choose a Markov partition $\{R_i\}_{i=0}^{r^*-1}$ such that $p \in R_0$. Hence, $\pi(\bar{0}) = p$. Note that the projection $\pi : \Sigma_A^+ \to \Lambda$ induces a map $\pi^* : C^0(\Lambda) \to C^0(\Sigma_A^+)$ by $\pi^* g = g \circ \pi \quad \forall g \in C^0(\Lambda)$.

Assume further that near p, there is a local coordinate system such that

$$fx = x\big(1 + |x|^r\big) + g(x), \tag{2.1}$$

where

$$|g(x)|, \quad |x| \ \|Dg(x)\|, \quad |x|^2 \ \|D^2 g(x)\| = O(|x|^{1+r'}) \tag{2.2}$$

for some $r' > r$ as x near p.

Lemma 2.1. *Suppose $f(x)$ satisfies (2.1) and (2.2). Then there is $0 < \delta < 1$, and an integer $k_0 \geq k_0'$ such that for all large k,*

$$\frac{1}{(k+k_0)r}\Big(1 - \frac{1}{(k+k_0)^\delta}\Big) \leq |x|^r \leq \frac{1}{(k+k_0')r}\Big(1 + \frac{1}{(k+k_0')^\delta}\Big) \quad \forall x \in P_k. \tag{2.3}$$

Proof. We may assume that $\Lambda \subset I$ so that we can drop the norm sign $|\cdot|$.

We claim that if $fx \leq x + x^{1+\gamma} + t_0 x^{1+r'}$ for some $t_0 > 0$, then there is $0 < \delta < 1$ such that for all large n, $x^r \leq \dfrac{1}{nr}\Big(1 - \dfrac{1}{n^\delta}\Big)$ implies

$$\big(fx\big)^r \leq \frac{1}{(n-1)r}\Big(1 - \frac{1}{(n-1)^\delta}\Big). \tag{2.4}$$

This implies the first inequality of (2.3). In fact, for any large k we can always find k_0 such that $x^r \geq \dfrac{1}{(k+k_0)r}\Big(1 - \dfrac{1}{(k+k_0)^\delta}\Big) \quad \forall x \in P_{k+k_0}$. Then we use induction.

Denote $\gamma_n = \gamma\big(1 - n^{-\delta}\big)^{-1}$. By the condition,

$$\big(fx\big)^\gamma \leq x^\gamma\big(1 + x^\gamma + t_0 \cdot x^{\gamma'}\big)^\gamma \leq \frac{1}{n\gamma_n}\Big(1 + \frac{1}{n\gamma_n} + \frac{t_0}{(n\gamma_n)^{\gamma'/\gamma}}\Big)^\gamma.$$

To prove the lemma we only need to show that

$$\frac{1}{n\gamma_n}\Big(1 + \frac{1}{n\gamma_n} + \frac{t_0}{(n\gamma_n)^{\gamma'/\gamma}}\Big)^\gamma \leq \frac{1}{(n-1)\gamma_{n-1}},$$

or, equivalently,

$$\frac{n-1}{n}\Big(1 + \frac{1}{n\gamma} - \frac{1}{n^{1+\delta}\gamma} + \frac{t_0}{(n\gamma_n)^{\gamma'/\gamma}}\Big)^\gamma \leq \frac{\gamma_n}{\gamma_{n-1}} = \frac{1-(n-1)^{-\delta}}{1-n^{-\delta}}. \tag{2.5}$$

Take $\delta < \min\{1, \gamma'/\gamma - 1\}$. Then $(n\gamma_n)^{-(\gamma'/\gamma)}$ is of higher order. It is easy to see that

$$\lim_{n\to\infty} n^{1+\delta}\Big(\frac{n-1}{n}\big(1 + \frac{1}{n\gamma} - \frac{1}{n^{1+\delta}\gamma}\big)^\gamma - 1\Big) = \lim_{n\to\infty} n^{1+\delta}\Big(-\frac{\gamma}{n^{1+\delta}\gamma} - \frac{\gamma+1}{2n^2\gamma}\Big) = -1$$

and

$$\lim_{n\to\infty} n^{1+\delta}\Big(\frac{1-(n-1)^{-\delta}}{1-n^{-\delta}} - 1\Big) = \lim_{n\to\infty} n \cdot \frac{1-(1-n^{-1})^{-\delta}}{1-n^{-\delta}} = -\delta.$$

So we know that as $n \to \infty$, the left side in (2.5) is like $1 - n^{-(1+\delta)}$ and the right side is like $1 - \delta n^{-(1+\delta)}$. Since $\delta < 1$, the right side is larger for all large n.

The second inequality in (2.3) can be proved similarly. $\square$

2.1. Maps on the unit interval. Let f be a piecewise smooth expanding map from the unit interval I *onto* itself with an indifferent fixed point $p = 0$. Denote by f' the derivative of f.

Theorem F. *Suppose $f : I \to I$ is an expanding map with an indifferent fixed point 0 that satisfies Assumptions B(I)-(III) with $\Lambda = I$, and near 0 f has the form (2.1) with $r > 0$. Then for any potential φ such that $\pi^*\varphi$ satisfies Assumptions A(I)-(III) with $\beta > 0$, or Assumptions A(I)-(III) and (III$'$) with $-1 < \beta \le \beta' \le 0$, Theorems A-E and their corollaries hold.*

In particular, if we take $\varphi(x) = -\log f'(x)$, then φ satisfies Assumptions A(I)-(III) and (III$'$) with $\alpha = \beta = \beta' = \gamma = r^{-1}$. In this case, the measure ν obtained in Theorem A is the Lebesgue measure, the measure μ is an absolutely continuous invariant measure, and the density function h satisfies $h^*(\bar{1}) = 0$. Moreover, μ is finite if $0 < r < 1$ and infinite if $1 \le r < \infty$.

Remark 2.1. If we take $\gamma = r^{-1}$, then the map $\pi : \Sigma_A^+ \to I$ is Lipschitz. So any Hölder potential on I satisfies the condition in the theorem.

Remark 2.2. The requirement of conditions (2.1) can be slightly relaxed. For example, the same proof can go through if we assume $x + ax^{1+r} \le fx \le x + bx^{1+r}$ for some $0 < a \le b$.

These systems with potential $\varphi(x) = -\log f'(x)$ have been studied extensively. The part concerning the existence of the absolutely continuous invariant measure μ is well known (see e.g. [Pi], [T]). It is proved that μ is a weak Gibbs measure (see e.g. [Yu2], [Yu3]) and an equilibrium (see e.g. [Yu1], [S1]).

The rates of convergence to the equilibriums and rates of decay of correlations are also well known for the case $r \in (0, 1)$ ([Y2], [H1], [S4], [G], also [LiSV], [PY]).

Proof of Theorem F. We only need to show that if $\varphi(x) = -\log f'(x)$, then it satisfies Assumptions A(I)-(III) and (III$'$). Let $\alpha = \beta = \beta' = \gamma = r^{-1}$.

Clearly, φ is continuous.

By (2.1) and (2.2), there exist $t_1, t_2 > 0$ such that

$$f'(x) \ge 1 + (1 + r)x^r - t_1 x^{r'},$$

$$f''(x) \le r(1 + r)x^{-1+r} + t_2 x^{-1+r'} \le C x^{-1+r}$$

for some $C \ge r(1 + r)$.

Let k be a large integer and $x \in P_k$. By the first inequality in (2.3), we have $x^r \ge (k + k_0)^{-1} r^{-1}$ and therefore

$$f''(x) \le C\big((k + k_0)r\big)^{\alpha(1-r)} \le C r^{\alpha-1}(k + k_0)^{\alpha-1} \le C_\varphi k^{\alpha-1} \tag{2.8}$$

for some $C_\varphi \ge C(2r)^{\alpha-1} > 0$ if $k \ge k_0$. Also, note that $f'(x)$ is bounded below by

$$1 + (1 + r)\frac{1}{(k + k_0)r}\Big(1 - \frac{1}{(k + k_0)^\delta}\Big) - t_1\Big(\frac{\beta}{k + k_0}\Big)^{\beta r'}\Big(1 - \frac{1}{(k + k_0)^\delta}\Big)^{\beta r'}.$$

Since $\beta r' \ge 1 + \delta$, we can find $C_\delta \ge C'_\delta > 0$ such that

$$f'(x) \ge 1 + \frac{1 + r}{(k + k_0)r} - \frac{C'_\delta}{(k + k_0)^{1+\delta}} \ge 1 + \frac{\beta + 1}{k} - \frac{C_\delta}{k^{1+\delta}}. \tag{2.9}$$

Hence, Assumptions A (II) and (III) follow from the definition of φ and the fact that $s > \log(1+s) \geq s - s^2/2$.

Assumption A (III′) can be obtained similarly. □

2.2. Parabolic rational maps. A rational map $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ on the Riemannian sphere $\bar{\mathbb{C}}$ with degree larger than or equal to 2 is *parabolic* if its Julia set $J = J(f)$ contains indifferent fixed points or periodic orbits but no critical point. The equivalent condition is that restricted to J, the map is positive expansive but not expanding in the spherical metric ([DU2]). The map has Markov partitions of arbitrarily small diameter ([DU3]).

In the case that an indifferent orbit contains more than one point, we can take f^n to get indifferent fixed points.

We say that a measure μ on J is a *measure of full Hausdorff dimension* if $\dim_H(\mu) = \dim_H(J)$, where $\dim_H(\mu) = \inf\{\dim_H(\Lambda_0) : \ \Lambda_0 \subset \Lambda, \mu(\Lambda_0) = 1\}$.

Theorem G. *Suppose $f : J \to J$ is the restriction to the Julia set of a parabolic rational map on the Riemannian sphere with an indifferent fixed point p and where f is topologically mixing. Then for any potential φ such that $\pi^*\varphi$ satisfies Assumptions A(I)-(III) with $\beta > 0$, or Assumptions A(I)-(III) and (III′) with $-1 < \beta \leq \beta' \leq 0$, Theorems A-E and their corollaries hold.*

Further, we suppose that near p, the Taylor expansion of f can be written as $f(z) = z + z^{1+r}$+higher order terms, and that t is the Hausdorff dimension of J. If we take $\varphi(x) = -t\log|f'(x)|$, then φ satisfies Assumptions A(I)-(III) and (III′) with $\alpha = \gamma = r^{-1}$ and $\beta = \beta' = t\big(1+r^{-1}\big)-1$. In this case, the conformal measure ν and invariant measure μ are measures of full Hausdorff dimension, and the density function h satisfies $h^*(\bar{1}) = 0$. Moreover, μ is finite if $2 < t\big(1+r^{-1}\big) < \infty$, and infinite if $1 < t\big(1+r^{-1}\big) \leq 2$.

For potentials of the form $\varphi(x) = -t\log|f'(x)|$, existence of conformal measures and invariant measures of such maps was proved, and some statistical properties such as central limit theorems and the wandering rates have been established by Denker, Urbański and Aaronson (see e.g. [DU2]-[DU4], [ADU] and their references). We refer to [U2], Section 3, for complete information on what is known before.

For general rational maps f on $\bar{\mathbb{C}}$ and general Hölder potential φ, it is known that if $P(f,\varphi) > \sup\{|\varphi(x)| : x \in J\}$, then all of the main results corresponding to $P(f,\varphi) > 0 = \varphi(\bar{0})$ in the above theorem hold (see [DU1], [Pr], [Ha]).

Remark 2.3. The Hausdorff dimension $\dim_H(J)$ of J is larger than $r/(1+r)$ (see e.g. [ADU]). So we always have $1 < \dim_H(J)\big(1+r^{-1}\big)$.

Remark 2.4. Note that here a conformal measure ν means an $e^{\log\lambda-\varphi}$-conformal measure. That is, ν satisfies (1.6). If $\lambda = 1$, then ν is the same conformal measure studied by Denker and Urbeński.

Proof of Theorem G. It is obvious that Assumptions B(I)-(III) are satisfied. So we only need consider the case $\varphi(x) = -t\log f'(x)$.

Assumption A(I) is clear. Assumption A(II) follows from the definition of φ and the same arguments for (2.8). For Assumption A(III), by the same arguments for

(2.9) we get that for $x \in P_k$,

$$-\varphi(x) \geq t \log\Big(1 + \frac{1+r^{-1}}{k} - \frac{C_\delta}{k^{1+\delta}}\Big) = \frac{t(1+r^{-1})}{k} - \frac{tC_\delta}{k^{1+\delta}} + O\Big(\frac{1}{k^2}\Big)$$

for some $C_\delta > 0$. Assumption A(III′) can be obtained in a similar way.

By Corollary C.1 and Theorem A, $P(\varphi, f) = \log \lambda \geq \varphi(0) = 0$. So by Theorem D,

$$0 \leq P(\varphi, f) = h_\mu(f) + \int \varphi d\mu = h_\mu(f) - t \int \log f' d\mu.$$

That is,

$$t \leq \frac{h_\mu(f)}{\int \log f' d\mu}.$$

Since μ is ergodic, and f is a conformal map, the right side of the equality is equal to $\dim_H(\mu)$. So we get $t \leq \dim_H(\mu)$ and therefore $\dim_H(J) = t \leq \dim_H(\mu)$. It means that μ is a measure of full Hausdorff dimension. Since $\mu \ll \nu$, ν is also a measure of full Hausdorff dimension. □

2.3. Parabolic Cantor sets. Let $f : I \to \mathbb{R}^+$ be a piecewise smooth expanding map.

Denote $\Lambda = \{x \in I : f^n x \in I \ \forall n \geq 0\}$. Clearly, $f\Lambda = \Lambda$ and $0 \in \Lambda$. If $f(I) = I$, then $\Lambda = I$, and it becomes the same case studied in Subsection 2.1. If $f(I) \supset I$, then Λ is a Cantor set topologically.

Let t be the Hausdorff dimension of Λ.

Theorem H. *Suppose $f|_\Lambda : \Lambda \to \Lambda$ is an expanding map with an indifferent fixed point 0 that satisfies Assumptions B(I)-(III), and near 0, f has the form (2.1) with $r > 0$. Then the same conclusions stated in Theorem G hold, including the case $\varphi(x) = -t \log f'(x)$, the choice of α, β, β' and γ, and the conditions for finiteness of μ.*

Parabolic Cantor sets were studied by M. Urbański (see [U1] and its reference). He obtained the existence of invariant measure and conformal measure of full Hausdorff dimension and investigated the equilibrium state and the phase transition of the systems.

Proof of Theorem H. It is the same as for the proof of Theorem G. □

2.4. Maps on higher dimensional spaces. We can generalise the results in Subsection 2.1 to a higher dimensional case. For a map f from the m-dimensional cube I^m to itself, we denote by $\det Df(x)$ the determinant of Df at x.

Theorem I. *Suppose $f : I^m \to I^m$ is an expanding map with an indifferent fixed point 0 that satisfies Assumptions B (I)-(III) with $\Lambda = I^m$, and near 0, f has the form (2.1) with $r > 0$. Then similar results stated in Theorem F hold with $\varphi(x) = -\log |\det Df(x)|$ and $\alpha = \gamma = r^{-1}$, $\beta = \beta' = mr^{-1}$. Also, μ is finite if $0 < r < m$ and infinite if $m \leq r < \infty$.*

Some expanding maps with indifferent fixed points in higher dimensional space are studied by M. Yuri ([Yu1]-[Yu4]), and the maps we discuss here also satisfy her assumptions, though she did not give these kinds of examples explicitly.

Remark 2.5. In these examples we require that near 0 the map f has about the same expanding rates along different radial directions. If f has two neutral directions along which f has different expanding rates, then Proposition 3.1 fails to hold and therefore Assumption A(II) cannot be true (see examples in [HV]).

Remark 2.6. We can also discuss the case that Λ is a fractal in I^m which has the form $\{(c,s) : c \in \Gamma, s \in S\}$ near the fixed point, where $\Gamma[0,\infty)$ is a parabolic Cantor set and and $S \subset S^{m-1}$ is a fractal.

Proof of Theorem I. We only need to verify $\beta = \beta' = mr^{-1}$. The rest proof is the same as for Theorem F.

In fact,

$$|\det Df(x)| = \big(1+|x|^r\big)^{m-1}\big(1+(1+r)|x|^r\big) + O(|x|^{r'}) = 1+(m+r)|x|^r + O(|x|^{r'}).$$

If $x \in P_k$, then by (2.3),

$$\begin{aligned}-\varphi(x) =& \log|\det Df(x)| = \log\Big[\big(1+(m+r)|x|^r\big) + O(|x|^{r'})\Big] \\ \geq& \frac{(m+r)}{(k+k_0)r} - \frac{C'_\delta}{(k+k_0)^{1+\delta}} \geq \frac{mr^{-1}+1}{k} - \frac{C_\delta}{k^{1+\delta}}.\end{aligned}$$

So we have $\beta = mr^{-1}$. $\beta' = mr^{-1}$ can be obtained similarly. □

3. The operator $\mathcal{L}_\varphi$: Proof of Theorem A

Proposition 3.1. *There is $J_\varphi > 0$ such that for all $J \geq J_\varphi$, the following holds:*

i) *For $x, y \in Q_k$,*

$$\big|\varphi(0x) - \varphi(0y)\big| + JD^\alpha_{k+1} d(0x, 0y)^\theta \leq JD^\alpha_k d(x,y)^\theta,$$

and if $s \neq 0$, then for $x, y \in \Sigma^+_A$,

$$\big|\varphi(sx) - \varphi(sy)\big| + JK^\alpha_0 d(sx, sy)^\theta \leq JK^\alpha_0 d(x,y)^\theta.$$

ii) *For $x, y \in Q_k$, $w = w_0 w_1 \cdots w_{n-1}$ with $wx, wy \in Q_m$,*

$$\big|S_n\varphi(wx) - S_n\varphi(wy)\big| + JD^\alpha_m d(wx, wy)^\theta \leq JD^\alpha_k d(x,y)^\theta,$$

and if $w_{n-1} \neq 0$, then for $x, y \in \Sigma^+_A$,

$$\big|S_n\varphi(wx) - S_n\varphi(wy)\big| + JD^\alpha_m d(wx, wy)^\theta \leq JK^\alpha_0 d(x,y)^\theta.$$

Proof. i) First we assume $k \geq K_0$ and $x, y \in Q_k$. It is easy to check by the definition that $d(0x, 0y) \leq k^{\gamma+1}(k+1)^{-(\gamma+1)}d(x,y)$. By Assumption A(II),

$$\begin{aligned}&\big|\varphi(0x) - \varphi(0y)\big| + J(k+1)^\alpha d(0x,0y)^\theta \\ \leq& C_\varphi (k+1)^{\alpha-1} d(0x,0y)^\theta + J(k+1)^\alpha d(0x,0y)^\theta \\ \leq& J \cdot \Big(C_\varphi \frac{k^{\theta(\gamma+1)-\alpha}}{J(k+1)^{\theta(\gamma+1)-\alpha+1}} + \frac{k^{\theta(\gamma+1)-\alpha}}{(k+1)^{\theta(\gamma+1)-\alpha}}\Big) k^\alpha d(x,y)^\theta.\end{aligned}$$

If J_φ is large enough and $J \geq J_\varphi$, then the quantity in the parentheses is less than 1.

If $k < K_0$ or $s \neq 0$, then we take $J_\varphi > 0$ such that $\big(C_\varphi K_0^\alpha + JK_0^\alpha\big)\kappa^\theta \leq JK_0^\alpha$ whenever $J \geq J_\varphi$. Therefore we can get the results in a similar way.

ii) It can be obtained from i) by induction. ☐

Recall that $\hat{g}$ is defined in (1.7).

Corollary 3.2. *Let $J \geq J_\varphi$ and $x = 0\bar{x}, y = 0\bar{y} \in \mathcal{O}_2$. If $g(s\bar{y}) \leq g(s\bar{x})e^{JK_0^\alpha d(s\bar{x},s\bar{y})^\theta}$ $\forall s \neq 0$, then $\hat{g}(y) \leq \hat{g}(x)e^{JK_0^\alpha d(x,y)^\theta}$.*

Proof. It follows from the above proposition and the fact that

$$\frac{\hat{g}(y)}{\hat{g}(x)} \leq \max_{s\neq 0}\Big\{\frac{e^{\varphi(s\bar{y})}g(s\bar{y})}{e^{\varphi(s\bar{x})}g(s\bar{x})}\Big\} \leq \max_{s\neq 0}\Big\{e^{\varphi(s\bar{y})-\varphi(s\bar{x})}e^{JK_0^\alpha d(s\bar{x},s\bar{y})^\theta}\Big\}.$$

By Proposition 3.1.i), the right side is less than or equal to $e^{JK_0^\alpha d(x,y)^\theta}$. ☐

Proof of Theorem A. By Lemma 3.3 there exist $\lambda^* \geq e^{\varphi(\bar{0})} = 1$, and $\nu \in \mathcal{M}(\Sigma_A^+)$, which is positive on nonempty open sets, such that $\mathcal{L}_\varphi^*\nu = \lambda^*\nu$.

Fix $J \geq J_\varphi$ and take a constant $J^* > J_\varphi$ large enough, which can be determined in the proof of Lemma 3.5. Let $\mathcal{B} = C^0\big(\Sigma_A^+\backslash\{\bar{0}\}\big)$ but with the norm

$$\|g\| = \sup_{x\in\Sigma_A^+\backslash\{\bar{0}\}}\big\{e^{-J^*k(x)}g(x)\big\}, \tag{3.1}$$

where $k(x) = k$ if $x \in P_k$. It is easy to check that $\mathcal{B}$ is a Banach space.

Lemma 3.4 below implies that $\mathcal{L}_\varphi : \mathcal{B} \to \mathcal{B}$ is continuous.

Take

$$\mathcal{H} = \{g \in \overline{\mathcal{G}}_J : \ g(\bar{1}) = 1\}.$$

$\mathcal{H}$ is not empty since it contains a constant function $g(x) = 1$. Clearly, $\mathcal{H}$ is a convex set. By Lemma 3.5, $\mathcal{H}$ is compact.

Define an operator $\bar{\mathcal{L}} : \mathcal{B} \to \mathcal{B}$ by $\bar{\mathcal{L}}g = \mathcal{L}_\varphi g/(\mathcal{L}_\varphi g)(\bar{1})$. $\bar{\mathcal{L}}$ is continuous because $\mathcal{L}_\varphi$ is continuous. By Lemma 3.6, $\bar{\mathcal{L}}\mathcal{H} \subset \mathcal{H}$. By the Schauder-Tychonoff Fixed Point Theorem (see e.g. [DS]), $\bar{\mathcal{L}}$ has a fixed point $h \in \mathcal{H}$. So we have $h \in \overline{\mathcal{G}}_J$, $\forall J \geq J_\varphi$, and $\mathcal{L}_\varphi h = \big((\mathcal{L}_\varphi h)(\bar{1})\big)h$. Denote $\lambda = (\mathcal{L}_\varphi h)(\bar{1})$.

By Lemma 3.7, $\lambda = \lambda^*$. Hence, we also have $\mathcal{L}_\varphi^*\nu = \lambda\nu$.

To prove that μ is σ-invariant, we can check directly that $\mathcal{L}_\varphi\big(h\cdot(g\circ\sigma)\big) = g\cdot(\mathcal{L}_\varphi h)$; then $\mu(g\circ\sigma) = \nu\big(h\cdot(g\circ\sigma)\big) = \nu\big(\lambda^{-1}\mathcal{L}_\varphi\big(h\cdot[g\circ\sigma]\big)\big) = \nu\big((\lambda^{-1}\mathcal{L}_\varphi h)\cdot g\big) = \nu(h\cdot g) = \mu(g)$. (See e.g. [B] for more details.)

The part $\lambda > 1$ if $-1 < \beta' \leq 0$ is proved in Lemma 4.5.

The last part of the theorem, concerning conditions under which μ is finite or infinite, follows from Corollary A.2 and the fact that $\Sigma_A^+ = \bigcup_{i=0}^\infty P_i$. ☐

Lemma 3.3. *There is a real number $\lambda^* \geq e^{\varphi(\bar{0})}$, and a measure $\nu \in \mathcal{M}(\Sigma_A^+)$, which is positive on nonempty open sets, such that $\mathcal{L}_\varphi^* \nu = \lambda^* \nu$.*

Proof. The map $\mu \to \mathcal{L}_\varphi^* \mu \big/ (\mathcal{L}_\varphi^* \mu)(1)$ is a continuous map from $\mathcal{M}(\Sigma_A^+)$ to itself. Since $\mathcal{M}(\Sigma_A^+)$ is compact in weak* topology, by the Schauder-Tychonoff Fixed Point Theorem the map has a fixed point ν. So if we take $\lambda^* = (\mathcal{L}_\varphi^* \mu)(1)$, then $\mathcal{L}_\varphi^* \nu = \lambda^* \nu$.

To prove $\nu U > 0$ for any open set U, it is enough to prove that for any word u, $\nu \mathcal{R}_u > 0$. Since σ is topologically mixing, $\sigma^n \mathcal{R}_u = \Sigma_A^+$ for some $n > 0$. Hence for any $x \in \Sigma_A^+$, there is an n-word v such that $vx \in \mathcal{R}_u$. We have

$$\mathcal{L}_\varphi^n \chi_{\mathcal{R}_u}(x) = \sum_{w \in \xi_n} e^{S_n \varphi(wx)} \chi_{\mathcal{R}_u}(wx) \geq e^{S_n \varphi(vx)} \chi_{\mathcal{R}_u}(vx) \geq e^{-n\|\varphi\|} > 0,$$

where ξ_n is defined in (1.1). So

$$\nu \mathcal{R}_u = \frac{1}{(\lambda^*)^n} (\mathcal{L}_\varphi^*)^n \nu(\chi_{\mathcal{R}_u}) \geq \frac{1}{(\lambda^*)^n} \nu(\mathcal{L}_\varphi^n \chi_{\mathcal{R}_u}) \geq \frac{\nu(e^{-n\|\varphi\|})}{(\lambda^*)^n} = \frac{1}{(\lambda^* e^{\|\varphi\|})^n} > 0.$$

Now we prove $\lambda^* \geq e^{\varphi(\bar{0})}$. Suppose $\lambda^* < e^{\varphi(\bar{0})}$. Since φ is a continuous function, there is $k > 0$ such that $(\lambda^*)^{-1} e^{\varphi(x)} > 1$ $\forall x \in \mathcal{O}_k$. Note that if $y \in P_{i+1}$, then $\sigma y \in P_i$. We get that for $i \geq k$,

$$\begin{aligned} \nu P_{i+1} &= \nu\big((\lambda^*)^{-1} \mathcal{L}_\varphi \chi_{P_{i+1}}\big) = \int \frac{1}{\lambda^*} \sum_{\sigma y = x} e^{\varphi(y)} \chi_{P_{i+1}}(y) d\nu(x) \\ &= \int \frac{1}{\lambda^*} e^{\varphi(0x)} \chi_{P_i}(x) d\nu(x) > \int \chi_{P_i}(x) d\nu(x) = \nu P_i. \end{aligned} \tag{3.2}$$

Hence, we have $\nu P_i > \nu P_k$ $\forall i > k$, contradicting finiteness of ν. $\square$

Lemma 3.4. *$\mathcal{L}_\varphi$ is a bounded linear operator.*

Proof. Take $g(x) = e^{J^* k(x)}$, where $k(x) = k$ if $x \in P_k$. Clearly g is the maximal element in the unit ball with respect to the norm in (3.1). Since $\mathcal{L}_\varphi$ is a positive operator, we only need to prove that $e^{-J^* k(x)} \mathcal{L}_\varphi g(x)$ is bounded.

Note that $k(0x) = k(x) + 1$ and $k(sx) = 0$ if $s \neq 0$. So

$$e^{-J^* k(x)} \mathcal{L}_\varphi g(x) = e^{-J^* k(x)} \Big(e^{\varphi(0x)} e^{J^* k(0x)} + \sum_{s \neq 0} e^{\varphi(sx)} e^{J^* k(sx)} \Big) \leq e^{\|\varphi\| + J^*} + r^* e^{\|\varphi\|},$$

where r^* is the number of different symbols used in Σ_A^+. $\square$

Lemma 3.5. *The set $\mathcal{H}$ is compact.*

Proof. For any $g \in \mathcal{H} \subset \overline{\mathcal{G}}_J$, we have

$$g(y) \leq g(x) e^{J D_k^\alpha d(x,y)^\theta} \leq g(x) e^{J k^{-\theta(1+\gamma)+\alpha}}$$

if $x \in P_k$, $y \in P'_k$ and $k > K_1$. So if $x \in P_k$, then we can choose $x^{(i)} \in P_{k-i}$, $i = 1, \cdots, k$, and $x^{(k)} = \bar{1}$. Denote $x = x^{(0)}$. By the above inequality we get

$$g(x) \le g(\bar{1}) \prod_{i=0}^{k-1} g(x^{(i)})/g(x^{(i+1)}) \le e^{\sum_{i=K_0}^{k-1} J(k-i)^{-\theta(1+\gamma)+\alpha}} \le e^{J^* k^{1-\theta(1+\gamma)+\alpha}} \tag{3.3}$$

for some $J^* > 0$ independent of g and $x^{(i)}$. Since $\alpha \le \theta(1+\gamma)$, $k^{1-\theta(1+\gamma)+\alpha} < k$. So $e^{-J^* k(x)} g(x) \to 0$ as $x \to \bar{0}$, and the convergence is uniform for all $g \in \mathcal{H}$.

Since $\mathcal{H} \subset \overline{\mathcal{G}}_J$, it is easy to see that $\mathcal{H}$ is uniformly bounded and equicontinuous outside $\mathcal{O}_k$ for any large k. With the above arguments we know that $\mathcal{H}$ is uniformly bounded and equicontinuous.

Clearly, $\mathcal{H}$ is closed in $\mathcal{B}$. We get the result. □

Lemma 3.6. *$\bar{\mathcal{L}}\mathcal{H} \subset \mathcal{H}$.*

Proof. Take $g \in \mathcal{H}$. We prove $\bar{\mathcal{L}}g \in \mathcal{H}$.

Take $x \in P_k$, $y \in P'_k$. Then

$$\frac{\bar{\mathcal{L}}g(y)}{\bar{\mathcal{L}}g(x)} = \frac{\sum\limits_{sy \in \Sigma_A^+} e^{\varphi(sy)} g(sy)}{\sum\limits_{sx \in \Sigma_A^+} e^{\varphi(sx)} g(sx)} \le \max_{sx, sy \in \Sigma_A^+} \Big\{ \frac{e^{\varphi(sy)} g(sy)}{e^{\varphi(sx)} g(sx)} \Big\}.$$

Note that $\dfrac{g(sy)}{g(sx)} \le e^{JK_0^\alpha d(sx,sy)^\theta}$ if $s \ne 0$ and $\dfrac{g(0y)}{g(0x)} \le e^{JD_{k+1}^\alpha d(0x,0y)^\theta}$. By Proposition 3.1, the right side of the above inequality is bounded by $e^{JD_k^\alpha d(x,y)^\theta}$. So $\bar{\mathcal{L}}g \in \overline{\mathcal{G}}_J$.

It is clear that $\bar{\mathcal{L}}_\varphi g > 0$ and $(\bar{\mathcal{L}}_\varphi g)(\bar{1}) = 1$. So $\bar{\mathcal{L}}g \in \mathcal{H}$. □

Lemma 3.7. *$\lambda = \lambda^*$.*

Proof. Define $h^{(n)}$ by

$$h^{(n)}(x) = \begin{cases} h(x) & \text{if } x \in Q_n; \\ 0 & \text{otherwise.} \end{cases}$$

We have $\mathcal{L}_\varphi h^{(n)} = \lambda h^{(n-1)} + \hat{h}\chi_{\mathcal{O}_n}$, where $\hat{h}(x)$ is defined in (1.7). So

$$\begin{aligned} \nu(h^{(n)}) =& (\lambda^*)^{-1} \mathcal{L}_\varphi^*(\nu)(h^{(n)}) = (\lambda^*)^{-1} \nu(\mathcal{L}_\varphi h^{(n)}) \\ =& (\lambda^*)^{-1} \lambda \nu(h^{(n-1)}) + (\lambda^*)^{-1} \nu(\hat{h}\chi_{\mathcal{O}_n}). \end{aligned}$$

Since $\nu(h^{(n)}) = \nu(h^{(n-1)}) + \nu(h\chi_{P_n})$, we have

$$\big(1 - (\lambda^*)^{-1}\lambda\big)\nu(h^{(n-1)}) = -\nu(h\chi_{P_n}) + (\lambda^*)^{-1}\nu(\hat{h}\chi_{\mathcal{O}_n}). \tag{3.4}$$

Note that $\nu(\hat{h}\chi_{\mathcal{O}_n}) \to 0$ as $n \to \infty$ since h is bounded on Q_0 and $\nu\mathcal{O}_n \to 0$. Now we prove $\nu(h\chi_{P_n}) \to 0$. This implies $\lambda = \lambda^*$, because $\nu(h^{(n-1)})$ increases with n.

Since $\nu(h\chi_{P_n}) = \mu P_n$ and μP_n decreases, the sequence $\{\nu(h\chi_{P_n})\}$ decreases with n. So if it does not converge to 0, then it is bounded away from 0. Hence by (3.4), $\nu(h\chi_{P_n})$ is roughly proportional to $\nu(h^{(n-1)})$. Since $\nu(h^{(n)}) = \nu(h^{(n-1)}) + \nu(h\chi_{P_n})$, it implies that $\nu(h^{(n)})$ and $\nu(h\chi_{P_n})$ increase exponentially fast. So restricted to P_n, $h(x)$ increases exponentially fast. It contradicts (3.3) which says that functions in $\overline{\mathcal{G}}_J$ increase subexponentially. □

4. The density function: Proof of Corollary A.1

Lemma 4.1. *For $x \neq \bar{0}$, $\lim\limits_{n\to\infty} h(0^n x) = \dfrac{\hat{h}(\bar{0})}{\lambda - 1}$ if $\lambda > 1$ and $\lim\limits_{n\to\infty} h(0^n x) = \infty$ if $\lambda = 1$.*

Proof. Since $\mathcal{L}_\varphi h = \lambda h$, we have $e^{\varphi(0^{n+1}x)} h(0^{n+1}x) + \hat{h}(0^{n+1}x) = \lambda h(0^n x)$, where $\hat{h}$ is defined in (1.7). Hence

$$h(0^{n+1}x) = e^{-\varphi(0^{n+1}x)}\big(\lambda h(0^n x) - \hat{h}(0^{n+1}x)\big), \tag{4.1}$$

or

$$\frac{h(0^{n+1}x)}{h(0^n x)} = e^{-\varphi(0^{n+1}x)}\Big(\lambda - \frac{\hat{h}(0^{n+1}x)}{h(0^n x)}\Big).$$

Since $h \in \bar{\mathcal{G}}_J$, $h(0^{n+1}x)/h(0^n x) \le e^{J_\varphi d(0^{n+1}x, 0^n x)} \to 1$ as $n \to \infty$. We also have $e^{-\varphi(0^{n+1}x)} \to e^{-\varphi(\bar{0})} = 1$ and $\hat{h}(0^{n+1}x) \to \hat{h}(\bar{0})$. The result of the lemma follows. □

Recall that h^* is defined in the statement of Theorem A.

Lemma 4.2. *Let $x \neq \bar{0}$.*

$$h(x) = \lambda^{-n} e^{S_n\varphi(0^n x)} h(0^n x) + \sum_{j=1}^{n} \sum_{s\neq 0} \lambda^{-j} e^{S_j\varphi(s0^{j-1}x)} h(s0^{j-1}x), \tag{4.2}$$

and if $\lambda = 1$, then $h^*(x) = \lim_{n\to\infty} e^{S_n\varphi(0^n x)} h(0^n x)$.

Proof. Note that for any x, $h(x) = \lambda^{-1} e^{\varphi(0x)} h(0x) + \sum_{s\neq 0} \lambda^{-1} e^{\varphi(sx)} h(sx)$. The equality holds by induction.

Let $\lambda = 1$. The sum in (4.2) increases with n. So we know that $e^{S_n\varphi(0^n x)} h(0^n x)$ decreases with n and therefore has a limit as $n \to \infty$. Then we use (1.5). □

Lemma 4.3. i) *$\lambda > 1$ if and only if $h^*(x) < 0$ for any $x \neq \bar{0}$.*

ii) *If $\lambda = 1$, then either $h^*(x) = 0$ for all $x \neq \bar{0}$, or $h^*(x) > 0$ for all $x \neq \bar{0}$.*

Proof. i) "⇒" If $\lambda > 1$, then $\lim\limits_{n\to\infty} \lambda^{-n} e^{S_n\varphi(0^n x)} h(0^n x) = 0$ because $h(0^n x)$ increases at most subexponentially, and $e^{S_n\varphi(0^n x)}$ decreases by Assumption A(III). So by (4.2), for any $x \neq \bar{0}$, $h^*(x) < 0$.

ii) Let $\lambda = 1$. By Proposition 3.1 and the fact $h \in \overline{\mathcal{G}}_J$, for any $x \in P_k$, $y \in P'_k$, $e^{S_n\varphi(0^n y)} h(0^n y)/e^{S_n\varphi(0^n x)} h(0^n x) \le e^{JK_0^\alpha d(x,y)^\theta} \le e^{JK_0^\alpha}$. So we know that the limit $\lim\limits_{n\to\infty} e^{S_n\varphi(0^n x)} h(0^n x)$ is either 0 or bounded away from 0 on any P_k, and therefore on any Q_k. Since $h^*(x)$ is equal to the limit, the result follows.

i) "⇐" If $h^*(x) < 0$ for some $x \neq \bar{0}$, then by part ii), $\lambda \neq 1$. So $\lambda > 1$. □

Lemma 4.4. *For any $k > 0$, there is $B_\varphi = B_{\varphi,k} > 0$ with $\lim_{k\to\infty} B_{\varphi,k} = 1$ such that for all $n > 0$, $x \in P_k$, $e^{S_n\varphi(0^n x)} \le B_\varphi\Big(\dfrac{k}{k+n}\Big)^{\beta+1}$.*

Suppose Assumption A (III′) also holds. Then for any $k > 0$, there is $B'_\varphi = B'_{\varphi,k} > 0$ with $\lim_{k\to\infty} B'_{\varphi,k} = 1$ such that for all $n > 0$, $x \in P_k$, $e^{S_n\varphi(0^n x)} \ge B'_\varphi\Big(\dfrac{k}{k+n}\Big)^{\beta'+1}$.

Moreover, if $\beta = \beta'$, then the limit $B^*_\varphi = \lim_{n\to\infty} n^{\beta+1} e^{S_n\varphi(0^n x)}$ exists.

Proof. By Assumption A (III), there exists $A'_\varphi, A_\varphi > 0$ such that for all $k > 0$, if $x \in P_k$, then

$$e^{\varphi(x)} \le 1 - \frac{\beta}{k+1} + \frac{A'_\varphi}{(k+1)^{1+\delta}} \le \Big(1 - \frac{1}{k+1}\Big)^{\beta+1}\Big(1 + \frac{A_\varphi}{(k+1)^{1+\delta}}\Big).$$

Taking product, we get

$$e^{S_n\varphi(0^n x)} \le \Big(\frac{k}{k+n}\Big)^{\beta+1} \prod_{i=k}^{k+n-1}\Big(1+\frac{A_\varphi}{(i+1)^{1+\delta}}\Big) \le \Big(\frac{k}{k+n}\Big)^{\beta+1} \prod_{i=k}^{\infty}\Big(1+\frac{A_\varphi}{(i+1)^{1+\delta}}\Big).$$

We let $B_{\varphi,k}$ be the product, which is convergent. Clearly, $\lim_{k\to\infty} B_{\varphi,k} = 1$.

The results corresponding to Assumption A (III′) can be obtained in a similar way.

Let $\beta = \beta'$. We know that the sequence $\{n^{\beta+1} e^{S_n\varphi(0^n x)}\}$ is bounded. Note that $e^{S_n\varphi(0^n x)} = e^{S_{n-k}\varphi(0^n x)} e^{S_k\varphi(0^k x)}$ is bounded between $B'_{\varphi,k}(k/n)^{\beta+1} e^{S_k\varphi(0^k x)}$ and $B_{\varphi,k}(k/n)^{\beta+1} e^{S_k\varphi(0^k x)}$. We have

$$B'_{\varphi,k} k^{\beta+1} e^{S_k\varphi(0^k x)} \le n^{\beta+1} e^{S_n\varphi(0^n x)} \le B_{\varphi,k} k^{\beta+1} e^{S_k\varphi(0^k x)}$$

for any $n \ge k$. Also, $\lim_{k\to\infty} B'_{\varphi,k} = \lim_{k\to\infty} B_{\varphi,k} = 1$. So $\{n^{\beta+1} e^{S_n\varphi(0^n x)}\}$ is a Cauchy sequence, and therefore is convergent. □

Lemma 4.5. *For any $k \ge 0$, there exist $B_\nu = B_{\nu,k} > 0$ and $C_\nu = C_{\nu,k} > 0$ such that the limits $\lim_{k\to\infty} B_{\nu,k}$ and $\lim_{k\to\infty} C_{\nu,k}$ exist, and for all $n \ge k$,*

$$\nu P_n \le \lambda^{-n} B_\nu n^{-(\beta+1)}, \qquad \nu \mathcal{O}_n \le \lambda^{-n} C_\nu n^{-\beta}.$$

Suppose Assumption A (III′) also holds. Then the above inequalities hold if we replace β, B_ν, C_ν and "$\le$" by β', B'_ν, C'_ν and "$\ge$", respectively. Hence, if $\beta' \le 0$, then $\lambda > 1$.

Moreover, if $\beta = \beta'$, then $B_{\nu,k}$ and $B'_{\nu,k}$ can be chosen in such a way that

$$\lim_{k\to\infty} B_{\nu,k} = \lim_{k\to\infty} B'_{\nu,k} = \lim_{n\to\infty} \lambda^n n^{\beta+1} \nu P_n = B^*_\nu. \tag{4.3}$$

Proof. By a similar method for (3.2), then by Proposition 3.1 and Lemma 4.4, for all large k, we have

$$\nu P_n = \frac{1}{\lambda^{n-k}} \int_{P_k} e^{S_{n-k}\varphi(0^{n-k}x)} d\nu(x) \le B_{\varphi,k} \frac{1}{\lambda^{n-k}} \left(\frac{k}{n}\right)^{\beta+1} \nu P_k. \tag{4.4}$$

So $\lambda^n n^{-(\beta+1)} \nu P_n$ is bounded. We can take $B_{\nu,k} = B_{\varphi,k} \max_{n\ge k}\{\lambda^n n^{-(\beta+1)}\nu P_n\}$; then we get the upper bound estimates for νP_n.

The estimate for $\nu \mathcal{O}_n$ follows from the fact that $\mathcal{O}_n = \bigcup_{i\ge n} P_i$.

If Assumption A (III′) also holds, the lower bound estimates can be made similarly. So if $\beta' \in (-1, 0]$, then $\lambda > 1$ since ν is a probability measure.

Let $\beta = \beta'$. The sequence $\{\lambda^n n^{-(\beta+1)}\nu P_n\}$ is bounded. By (4.4) we have

$$B'_{\nu,k}\lambda^k k^{-(\beta+1)}\nu P_k \le \lambda^n n^{-(\beta+1)}\nu P_n \le B_{\nu,k}\lambda^k k^{-(\beta+1)}\nu P_k$$

for any $n \ge k$. Also, $\lim_{k\to\infty} B'_{\varphi,k} = \lim_{k\to\infty} B_{\varphi,k} = 1$. So $\{\lambda^n n^{-(\beta+1)}\nu P_n\}$ is a Cauchy sequence, and therefore the limits in (4.3) exist. □

The next lemma is for the proof of uniqueness of equilibriums in Theorem D and phase transition in Corollary D.1.

Lemma 4.6. i) *If $h^*(\bar{1}) > 0$, then there exists a continuous function $\varphi^+ > \varphi$ on $\Sigma_A^+ \backslash \{\bar{0}\}$ such that for any $\varphi^+ \ge \varphi' \ge \varphi$ satisfying Assumptions (I)-(III), $h'^*(\bar{1}) > 0$, where h' is the density function given in Theorem A for φ'.*

If Assumption A(III′) also holds with $\beta' = \beta$, then φ^+ can be taken in such a way that $\varphi^+(x) - \varphi(x) > a/k$ for some $a > 0$ as $x \in P_k$ and for all large k.

ii) If there exists $\varphi' \ge \varphi$ with $\varphi' \ne \varphi$ that satisfies Assumptions (I)-(III) such that $h'^*(\bar{1}) \ge 0$, then $h^*(\bar{1}) \ge 0$.

Proof. i) From the proof of Lemma 4.3 we know that if $h^*(\bar{1}) > 0$, then $h^*(x) \ge c > 0$ for some c as $x \in P_0$. By the definition of h^* given in (1.5) and $\tilde{\mathcal{L}}_{\tilde{\varphi}}$ given in (1.4) we know that $h^*(\bar{1}) > 0$ implies $h(x) > \tilde{\mathcal{L}}_{\tilde{\varphi}} h(x) + c$ for any $x \in P_0$. So by continuity we can find $\varphi^+ > \varphi$ such that for any $\varphi^+ \ge \varphi' \ge \varphi$, $h(x) > \tilde{\mathcal{L}}_{\tilde{\varphi}'} h(x)$.

Take φ that satisfies Assumptions (I)-(III), and $\varphi^+ \ge \varphi' \ge \varphi$. Note that $\tilde{\mathcal{L}}_{\tilde{\varphi}'}$ maps the set of continuous functions on P_0 to itself. So by the similar method as in the proof of Lemma 3.3, we get that there is a conformal measure ν' on P_0 and a constant λ' such that $\tilde{\mathcal{L}}^*_{\tilde{\varphi}'}\nu' = \lambda'\nu'$. Hence,

$$\lambda'\nu'(h) = \big(\tilde{\mathcal{L}}^*_{\tilde{\varphi}'}\nu'\big)(h) = \nu'(\tilde{\mathcal{L}}_{\tilde{\varphi}'} h) < \nu'(h).$$

That is, $\lambda' < 1$.

Let h' be the density function obtained in Theorem A for φ'. We claim $h'^*(\bar{1}) > 1$. In fact, if not, then by Lemma 4.3 we have $h'(x) \le \tilde{\mathcal{L}}_{\tilde{\varphi}'} h'(x)$ for all $x \in P_0$. Then the same argument gives $\lambda'\nu'(h') \ge \nu'(h)$. It implies $\lambda' \ge 1$, a contradiction.

Now we assume that Assumption A(III′) also holds and $\beta = \beta'$. By Lemma 4.3, $h'^*(\bar{1}) > 1$ implies $\lambda = 1$, and then by Lemma 4.5 we have $\beta = \beta' > 0$. Take

$\varphi_1^+ \geq \varphi$ such that $\varphi_1^+ > \varphi$ on Q_{K_1}, where K_1 is given in Assumption (III), and such that

$$\sum_{j=1}^{\infty}\sum_{s\neq 0} e^{S_j\varphi_1^+(s0^{j-1}x)}h(s0^{j-1}x) - \sum_{j=1}^{\infty}\sum_{s\neq 0} e^{S_j\varphi(s0^{j-1}x)}h(s0^{j-1}x) \leq c/2.$$

This is possible; for example, we can let $\varphi_1^+ = \varphi$ on $\mathcal{O}_{K_1+1}$. By Lemma 4.4 we know that there exists $B > 0$ such that $e^{S_j\varphi_1^+(s0^{j-1}x)} \leq Bj^{-(1+\beta)}$ for all $j > 0$ and $x \in P_0$. So if we take $\varphi_a(x) = \varphi_1^+(x) + a/k$ for all $x \in P_k$ and $k \geq K_1$, and $\varphi_a(x) = \varphi_1^+(x)$ for all $x \in Q_{K_1}$, then

$$\sum_{j=K_1}^{\infty}\sum_{s\neq 0} e^{S_j\varphi_a(s0^{j-1}x)}h(s0^{j-1}x) \to \sum_{j=K_1}^{\infty}\sum_{s\neq 0} e^{S_j\varphi_1^+(s0^{j-1}x)}h(s0^{j-1}x),$$

as $a \to 0$. Hence we can take $a > 0$ small enough such that the difference is less than $c/2$. Therefore $\varphi^+ = \varphi_a$ is the function we need.

ii) It can be proved in a similar way. $\square$

The next two lemmas are for the case $h^*(\bar{1}) = 0$.

Lemma 4.7. *Suppose $h^*(\bar{1}) = 0$. Then there exists $H > 0$ such that for all $k > 0$, $h(x) \leq Hk$ if $x \in P_k$.*

Proof. Since $h^*(\bar{1}) = 0$, we have that for any $x \neq \bar{0}$,

$$h(x) = \sum_{j=1}^{\infty}\sum_{s\neq 0} e^{S_j\varphi(s0^{j-1}x)}h(s0^{j-1}x).$$

Since h is bounded on P_0, we denote by H_0 the upper bound. Also, $e^{S_j\varphi(s0^{j-1}x)} = e^{\varphi(s0^{j-1}x)}e^{S_{j-1}\varphi(0^{j-1}x)}$. So if $x \in P_k$, then by Lemma 4.4, we get

$$h(x) \leq r^* e^{\|\varphi\|} H_0 \sum_{j=1}^{\infty} e^{S_{j-1}\varphi(0^{j-1}x)} \leq r^* e^{\|\varphi\|} H_0 B_\varphi \sum_{j=1}^{\infty}\Big(\frac{k}{k+j-1}\Big)^{\beta+1} \leq Hk$$

for some $H > 0$. $\square$

Lemma 4.8. *Suppose $h^*(\bar{1}) = 0$. Then there is $C_h > 0$ such that for all $x \in P_0$, for all large n,*

$$\frac{\beta}{n} h(0^n x) \leq \hat{h}(\bar{0})\Big(1 + \frac{C_h}{n^\delta}\Big). \tag{4.5}$$

If φ also satisfies Assumption A(III′) and $h^*(\bar{1}) = 0$, then there is $C_h' > 0$ such that for all $x \in P_0$, for all large n,

$$\frac{\beta'}{n}\ h(0^n x) \geq \hat{h}(\bar{0})\Big(1 - \frac{C_h'}{n^\delta}\Big). \tag{4.6}$$

Proof. We only prove (4.5). The inequality (4.6) can be proved similarly.

By Lemma 4.3, we have $\lambda = 1$ and therefore by Theorem A, $\beta > 0$.

Recall that we assume $\delta \le \min\{1, \gamma\theta\}$ after Assumption A(III′) is stated.

Take $C_h > 0$ such that for all large n,

$$\frac{\beta n - 2 - \beta}{2(n+1)^2} \cdot \frac{C_h}{(n+1)^\delta} - \frac{(1+\beta)^2}{(n+1)^2} - \frac{n+\beta+2}{(n+1)^2} \cdot \frac{2\beta J_0 C_\gamma}{n^{\gamma\theta}} - \frac{C_\delta (n + \epsilon n - \beta)}{(n+1)^{2+\delta}} > 0,$$

where $J_0 = J_\varphi K_0^\alpha$.

First we claim that if there is $x \in P_0$ such that for $\epsilon \ge C_h n^{-\delta}$,

$$\frac{\beta}{n} h(0^n x) \ge \hat{h}(\bar{0})\big(1+\epsilon\big) \tag{4.7}$$

holds for some large n, then

$$\frac{\beta}{n+1} h(0^{n+1} x) \ge \hat{h}(\bar{0})\Big(1 + \epsilon + \frac{\beta}{2(n+1)}\epsilon\Big). \tag{4.8}$$

Now we prove the claim. By Lemma 3.3 and Assumption A(III),

$$e^{\varphi(\bar{0}) - \varphi(0^{n+1}x)} \ge 1 + \frac{\beta+1}{n+1} - \frac{C_\delta}{(n+1)^{1+\delta}}. \tag{4.9}$$

Note that $h \in \overline{\mathcal{G}}_{J_\varphi}$. By Corollary 3.2 and (1.2), if n is large enough, then

$$\hat{h}(0^{n+1}x) \le \hat{h}(\bar{0}) e^{J_0 d(0^n x, \bar{0})^\theta} \le \hat{h}(\bar{0}) e^{J_0 C_\gamma n^{-\gamma\theta}} \le \hat{h}(\bar{0})(1 + 2J_0 C_\gamma n^{-\gamma\theta}). \tag{4.10}$$

So by (4.7) and (4.10), if n is large enough, then

$$\begin{aligned}\frac{1}{\hat{h}(\bar{0})}\Big(\lambda h(0^n x) - \hat{h}(0^{n+1}x)\Big) \ge& \frac{n}{\beta}\big(1+\epsilon\big) - \big(1 + \frac{2J_0 C_\gamma}{n^{\gamma\theta}}\big) \\ =& \frac{n-\beta}{\beta} + \frac{n}{\beta}\epsilon - \frac{2J_0C_\gamma}{n^{\gamma\theta}} \ge \frac{n-2\beta}{\beta}. \end{aligned} \tag{4.11}$$

By (4.1) and then by (4.9) and (4.11), we have

$$\begin{aligned}&\frac{\beta}{n+1}\frac{h(0^{n+1}x)}{\hat{h}(\bar{0})} \ge \frac{\beta}{n+1} \cdot e^{-\varphi(0^{n+1}x)} \cdot \frac{1}{\hat{h}(\bar{0})}\big(h(0^n x) - \hat{h}(0^{n+1}x)\big) \\ \ge& \frac{\beta}{n+1} \cdot \frac{n+\beta+2}{n+1} \cdot \Big[\frac{n-\beta}{\beta} + \frac{n}{\beta}\epsilon - \frac{2J_0C_\gamma}{n^{\gamma\theta}}\Big] - \frac{\beta}{n+1} \cdot \frac{C_\delta}{(n+1)^{1+\delta}} \cdot \frac{n + \epsilon n - 2\beta}{\beta} \\ =& \frac{(n+\beta+2)(n-\beta)}{(n+1)^2} + \frac{(n+\beta+2)n}{(n+1)^2}\epsilon - \frac{n+\beta+2}{(n+1)^2} \cdot \frac{2\beta J_0 C_\gamma}{n^{\gamma\theta}} - \frac{C_\delta(n+\epsilon n - 2\beta)}{(n+1)^{2+\delta}}.\end{aligned}$$

Note that

$$\frac{(n+\beta+2)(n-\beta)}{(n+1)^2} = 1 - \frac{(1+\beta)^2}{(n+1)^2},$$

$$\frac{(n+\beta+2)n}{(n+1)^2}\epsilon = \epsilon + \frac{\beta}{2(n+1)}\epsilon + \frac{\beta n - 2 - \beta}{2(n+1)^2}\epsilon.$$

Also note that $\epsilon \geq C_h n^{-\delta}$. By the choice of C_h, we get (4.8). The claim is true.

Using this claim we can get the result of the lemma. Otherwise we have an $\epsilon > C_h n_0^{-\delta}$ such that (4.7) holds for some large n_0. Then using the claim repeatedly, we get

$$\frac{\beta}{n_0+k} h(0^{n_0+k}x) \geq \hat{h}(\bar{0})\Big(1+\epsilon+\epsilon \sum_{i=n_0}^{n_0+k-1} \frac{\beta}{2(i+1)}\Big) \qquad \forall k \geq 0.$$

Since the summation goes to infinity as $k \to \infty$, it contradicts the fact given by Lemma 4.7 that $h(0^n x) \leq Hn$ for all $n > 0$. □

Proof of Corollary A.1. The results for the case $h^*(\bar{1}) < 0$ and $h^*(\bar{1}) = 0$ are given in Lemma 4.1 and Lemma 4.8, respectively.

For the case $h^*(\bar{1}) > 0$, by Lemma 4.2, $\lim_{n\to\infty} e^{S_n\varphi(0^n x)} h(0^n x) = h^*(x)$, and by Lemma 4.4, $n^{-(\beta+1)} e^{-S_n\varphi(0^n x)} \geq B_{\varphi,0}^{-1}$. So the first part of the case follows. The rest can be proved similarly.

5. The function ψ and measure μ: Proof of Corollaries A.2 and A.3

Lemma 5.1. *Let w be an n-word, $n > 0$. Then for any integrable function g,*

$$\int e^{S_n\psi(wx)} g(wx) d\mu(x) = \int_{\mathcal{R}_w} g(x) d\mu(x).$$

Hence, if we take $g = \chi_{\mathcal{R}_{wu}}$ for any word u, then $\displaystyle\int_{\mathcal{R}_u} e^{S_n\psi(wx)} d\mu(x) = \mu\mathcal{R}_{wu}$.

Proof. Define $g^*(x) = g(x)$ if $x \in \mathcal{R}_w$ and $g^*(x) = 0$ otherwise. Then we have $e^{S_n\psi(wx)} g(wx) = \mathcal{L}_\psi^n g^*(x)$ for any x. So we get

$$\int e^{S_n\psi(wx)} g(wx) d\mu(x) = \mu(\mathcal{L}_\psi^n g^*) = \mu(g^*) = \int_{\mathcal{R}_w} g(x) d\mu(x).$$

By using $\chi_{\mathcal{R}_{wu}}(wx) = \chi_{\mathcal{R}_u}(x)$, we can get the second part of the lemma. □

Lemma 5.2. $e^{\psi(0x)} = \dfrac{e^{\varphi(0x)} h(0x)}{\lambda h(x)} = 1 - \dfrac{1}{\lambda h(x)} \hat{h}(0x)$. *In particular,* $e^{\psi(\bar{0})} = \dfrac{1}{\lambda}$.

Proof. Since $\mathcal{L}_\psi h(x) = \lambda h(x)$, we have $e^{\varphi(0x)} h(0x) + \hat{h}(0x) = \lambda h(x)$. So by the definition of ψ,

$$e^{\psi(0x)} = \frac{e^{\varphi(0x)} h(0x)}{\lambda h(x)} = 1 - \frac{1}{\lambda h(x)} \hat{h}(0x). \tag{5.1}$$

For the case $x = \bar{0}$, the result follows from the definition of ψ and (1.8). □

Lemma 5.3. *There exists $B_\delta > 0$ such that for all large n, the following hold:*

i) *if $h^*(\bar{1}) < 0$, then $e^{\psi(0x)} \le \frac{1}{\lambda} \cdot \Big(1 + \frac{B_\delta}{(n+1)^{\min\{1, \alpha-\theta(1+\gamma)\}}}\Big)$ $\forall x \in P_n$;*

ii) *if $h^*(\bar{1}) = 0$, then $e^{\psi(0x)} \le 1 - \frac{\beta}{n} + \frac{B_\delta}{n^{1+\delta}}$ $\forall x \in P_n$;*

iii) *if $h^*(\bar{1}) > 0$, then $e^{\psi(0x)} \ge 1 - \frac{B_\delta}{n^{1+\beta}}$ $\forall x \in P_n$.*

Suppose Assumption A (III′) also holds. Then there exists $B'_\delta > 0$ such that for all large n, the above inequalities hold if we interchange "$\le$" and "$\ge$", and then replace B_ψ and β by $-B'_\psi$ and β' in i) and ii), and by B'_ψ and β' in iii).

Proof. i) Note that by Assumption A (III), $\varphi(0x) \le -\frac{\beta+1}{n+2} + \frac{C_\delta}{(n+2)^{1+\delta}}$. Also since $h \in \overline{\mathcal{G}}_{J_\varphi}$, $h(0x)/h(x) \le e^{J_\varphi n^\alpha d(x,0x)^\theta} \le e^{J_\varphi (n+1)^{\alpha-\theta(1+\gamma)}}$. So if $h^*(\bar{1}) < 0$, the result can be obtained from the first equality of (5.1).

ii) By Lemma 4.8 and a similar method for (4.10), we have that for all large n,

$$\frac{1}{\lambda h(x)} \cdot \hat{h}(0x) \ge \frac{\beta}{n\hat{h}(\bar{0})}\Big(1 + \frac{C_h}{n^\delta}\Big)^{-1} \cdot \hat{h}(\bar{0})\Big(1 - \frac{2C_\gamma J_\varphi K_0^\alpha}{n^{\gamma\theta}}\Big) \ge \frac{\beta}{n} - \frac{B_\delta}{n^{1+\delta}}$$

for some large B_δ. Now the result follows from the second equality of (5.1).

iii) Since $h^*(\bar{1}) > 0$, by Lemma 4.2, $h(x)$ is of the same order as $e^{-S_n\varphi(x)}$. By Lemma 4.3, if $x = 0^n y \in P_n$, $y \in P_0$, then $e^{-S_n\varphi(x)} \ge B_\varphi^{-1}(n+1)^{\beta+1}$. Now we use the second equality of (5.1).

The other direction of the inequalities can be estimated in a similar way. □

Lemma 5.4. *For any $k > 0$, there is $C_\psi = C_{\psi,k} > 0$ and $C'_\psi = C'_{\psi,k} > 0$ with $\lim_{k\to\infty} C_{\psi,k} = \lim_{k\to\infty} C'_{\psi,k} = 1$ such that for all $n > 0$, $x \in P_k$, the following holds:*

i) *if $h^*(\bar{1}) < 0$, then $e^{S_n\psi(0^n x)} \le C_\psi \lambda^{-n}\Big(\frac{k}{k+n}\Big)^{\beta+1}$;*

ii) *if $h^*(\bar{1}) = 0$, then $e^{S_n\psi(0^n x)} \le C_\psi\Big(\frac{k}{k+n}\Big)^{\beta}$;*

iii) *if $h^*(\bar{1}) > 0$, then $C'_\psi h^*(x) \le e^{S_n\psi(0^n x)} \le C_\psi h^*(x)$.*

Suppose Assumption A (III′) also holds. Then i) and ii) are true if we replace "$\le$", β and C_ψ by "$\ge$", β' and C'_ψ, respectively.

Proof. i) Use Lemma 4.4 and the fact that both $h(0^n x), h(x) \to \lambda^{-1}$ as $k \to \infty$.

ii) It can be proved by using Lemma 5.3 and the same methods as for the proof of Lemma 4.4.

iii) Since $x \in P_k$, we know that $x = 0^k y$ for some $y \in P_0$. By the definition of ψ, we have

$$e^{S_n\psi(0^n x)} = e^{S_n\varphi(0^n x)}\frac{h(0^n x)}{h(x)} = \frac{e^{S_{n+k}\varphi(0^{n+k}y)}h(0^{n+k}y)}{e^{S_k\varphi(0^k y)}h(0^k y)}.$$

Then we take $C_{\psi,k} = \max_{y\in P_0}\sup_{n\ge 0}\Big\{\frac{e^{S_{n+k}\varphi(0^{n+k}y)}h(0^{n+k}y)}{e^{S_k\varphi(0^k y)}h(0^k y)}\Big\}$ and $C'_{\psi,k} = C^{-1}_{\psi,k}$. By Lemma 4.2, $C_{\psi,k}, C'_{\psi,k} \to 1$ as $k \to \infty$.

The other direction of the inequalities for i) and ii) can be obtained similarly. $\square$

Proof of Corollary A.2. Note that by the definition of μ, $\mu P_k = \nu(h\chi_{P_k}) \le \nu P_k \max\{h(x) : x \in P_k\}$. Also note that $\mathcal{O}_n = \bigcup_{i\ge n} P_i$. Hence, part i) follows from Corollary A.1 and Lemma 4.5 with $B_{\mu,k} = B_{\nu,k}\max\{h(x) : x \in P_k\}$, and part ii) follows from Lemma 4.8 and 4.5 with $B_{\mu,k} = B_{\nu,k}\beta^{-1}\hat{h}(\bar{0})(1 + C_h n^{-\delta})$.

By Lemma 5.1, $\mu P_{n+k} = \int_{P_k} e^{S_n\psi(0^n x)} d\mu(x)$. So we use part iii) of Lemma 5.4 to get $\mu P_{n+k} \ge C'_\psi \mu P_k \min\{h^*(x) : x \in P_k\} > 0$. It means that $\{\mu P_n\}$ are bounded away from 0. Since this is a decreasing sequence, we get part iii).

If Assumption A (III′) also holds, the lower bounds can be estimated similarly.

If $\beta' = \beta$, the existence of limits follows from (4.3) and the choice of these constants. $\square$

Proof of Corollary A.3. (I). It follows from continuity of φ on Σ_A^+ and h on $\Sigma_A^+\backslash\{\bar{0}\}$.

(II). Take an n-word $w = w_0w_1\cdots w_{n-1}$, and suppose $wx, wy \in P_m$. We have

$$S_n\psi(wy) - S_n\psi(wx) = S_n\varphi(wy) - S_n\varphi(wx) + \log\frac{h(wy)}{h(wx)} + \log\frac{h(x)}{h(y)}.$$

Since $h \in \mathcal{H}_{J_\varphi}$, $\log\dfrac{h(wy)}{h(wx)} \le J_\varphi D_m^\alpha d(wx, wy)^\theta$ and $\log\dfrac{h(x)}{h(y)} \le J_\varphi D_k^\alpha d(x, y)^\theta$. By Proposition 3.1 ii), we get the first part of (II) with $J_\psi = 2J_\varphi$.

For the second part, we write

$$\begin{aligned}\big(S_n\psi(wy) + \log h(y)\big) - &\big(S_n\psi(wx) + \log h(x)\big)\\ &= S_n\varphi(wy) - S_n\varphi(wx) + \log h(wy) - \log h(wx)\end{aligned}$$

and then use the second part of Proposition 3.1 ii).

(III) & (III′). The results follow from Lemma 5.3. $\square$

6. Exactness: Proof of Theorem B

Recall that $\mathcal{L}_\psi$ is defined in Section 1, after Corollary D.2 is stated.

Lemma 6.1. i) *$\mathcal{L}_\psi c = c$ for any constant function c.*

ii) *$\mu(\mathcal{L}_\psi g) = \mu(g)$ for any integrable function g.*

iii) *$\mu(|\mathcal{L}_\psi g|) \le \mu(|g|)$ for any function g in $L^1(\Sigma_A^+, \mu)$. Further, if g is continuous, and there exists $x \in \Sigma_A^+$, and n-words u and v such that $g(ux) > 0$ and $g(vx) < 0$, then $\mu(|\mathcal{L}_\psi^n g|) < \mu(|g|)$.*

Proof. i) Since $\mathcal{L}_\varphi h = \lambda h$, we get $\mathcal{L}_\psi c = \dfrac{1}{\lambda h}\mathcal{L}_\varphi(ch) = \dfrac{c}{\lambda h}\mathcal{L}_\varphi h = c$.

ii) This is because $\mu(\mathcal{L}_\psi g) = \nu\big(h\cdot\dfrac{1}{\lambda h}\mathcal{L}_\varphi(hg)\big) = \nu\big(\dfrac{1}{\lambda}\mathcal{L}_\varphi(hg)\big) = \nu\big(hg\big) = \mu(g)$.

iii) It is easy to check by using part i) that for any x, $|\mathcal{L}_\psi g(x)| \le \mathcal{L}_\psi|g(x)|$. Hence, by part ii), $\mu(|\mathcal{L}_\psi g|) \le \mu(\mathcal{L}_\psi|g|) = \mu(|g|)$.

For the second part, we have

$$\big|\mathcal{L}_\psi^l g(x)\big| = \Big|\sum_{w\in\xi_l} e^{\psi(wx)}g(wx)\Big| < \sum_{w\in\xi_l} e^{\psi(wx)}\big|g(wx)\big| = \big(\mathcal{L}_\psi^l|g|\big)(x)$$

for x given in the lemma. So we have $\mu(|\mathcal{L}_\psi^l g|) < \mu(\mathcal{L}_\psi^l |g|) = \mu(|g|)$. $\square$

For any x, we denote

$$\bar{g}(0x) = \frac{\sum_{s\neq 0} e^{\psi(sx)} g(sx)}{\sum_{s\neq 0} e^{\psi(sx)}} = \frac{\sum_{s\neq 0} e^{\varphi(sx)} h(s\tilde{x}) g(sx)}{\sum_{s\neq 0} e^{\varphi(sx)} h(sx)}. \tag{6.1}$$

That is, $\bar{g}(0x)$ is the average of $g(sx)$, $s \neq 0$, with weights $e^{\varphi(sx)}$. Since $\sum_{s\neq 0} e^{\psi(sx)} = 1 - e^{\psi(0x)}$, we have

$$\mathcal{L}_\psi g(x) = e^{\psi(0x)} g(0x) + (1 - e^{\psi(0x)}) \bar{g}(0x). \tag{6.2}$$

Lemma 6.2. *For any $x \in \Sigma_A^+$,*

$$\mathcal{L}_\psi^n g(x) = g(0^n x) e^{S_n \psi(0^n x)} + \sum_{j=1}^n (\overline{\mathcal{L}_\psi^{n-j} g})(0^j x)\big(1 - e^{\psi(0^j x)}\big) e^{S_{j-1}\psi(0^{j-1}x)}.$$

Proof. It can be obtained by (6.2) and induction. $\square$

Lemma 6.3. *For any continuous function g on Σ_A^+ with $\mu(|g|) < \infty$ and $\mu(g) = 0$,*

$$\lim_{n\to\infty} \mathcal{L}_\psi^n g(x) = 0 \qquad \forall x \in \Sigma_A^+ \setminus \{\bar{0}\},$$

and the convergence is in $L^1(\mu)$ and uniform on Q_k for any $k \geq 0$. Also,

$$\lim_{n\to\infty} \mathcal{L}_\psi^n g(\bar{0}) = \begin{cases} 0 & \text{if } \lambda > 1; \\ g(\bar{0}) & \text{if } \lambda = 1. \end{cases}$$

Proof. Let $x \in P_i$, $y \in P_i'$. By Corollary A.3(II), for any $n \geq 0$ we have

$$\begin{aligned}
&\big|\mathcal{L}_\psi^n g(x) - \mathcal{L}_\psi^n g(y)\big| \\
\leq &\sum_{w\in\xi_n} e^{S_n\psi(wx)} \big|g(wx) - g(wy)\big| + \sum_{w\in\xi_n} g(wy) e^{S_n\psi(wy)} \big|e^{S_n\psi(wx) - S_n\psi(wy)} - 1\big| \\
\leq &\operatorname{var}_n(g) \sum_{w\in\xi_n} e^{S_n\psi(wx)} + (e^{J_\psi D_i^\alpha d(x,y)^\theta} - 1) \sum_{w\in\xi_n} g(wy) e^{S_n\psi(wy)} \\
\leq &\operatorname{var}_n(g) + (e^{J_\psi D_i^\alpha d(x,y)^\theta} - 1) |\mathcal{L}_\psi^n g(y)|,
\end{aligned}$$

where $\operatorname{var}_n(g)$ is defined in (1.3). Note that $e^{J_\psi D_i^\alpha d(x,y)^\theta} - 1 \to 0$ as $d(x,y) \to 0$, and the convergence is uniform for all $x, y \in Q_k$ whenever $k > 0$ is fixed. Since $\mathcal{L}_\psi g(x)$ is the average value of the $g(sx)$ with weight $e^{\psi(sx)}$, $\{\mathcal{L}_\psi^n g : \ n \geq 0\}$ is uniformly bounded. The above arguments says that restricted to Q_k, $\{\mathcal{L}_\psi^n g : \ n \geq 0\}$ is equicontinuous. So the closure of $\{\mathcal{L}_\psi^n g : \ n \geq 0\}$ is compact. Therefore, there is a subsequence $\{n_i\}$ and a continuous function $g^{(k)}$ on Q_k such that $\mathcal{L}_\psi^{n_i} g \to g^{(k)}$. Since $Q_k \subset Q_{k+1}$ and $\bigcup_{k\ge 0} Q_k = \Sigma_A^+\backslash\{\bar 0\}$, by applying the diagonalization method, we know that there is a subsequence $\{n_i\}$ and a continuous function g^* on $\Sigma_A^+\setminus\{\bar 0\}$ such that $\mathcal{L}_\psi^{n_i} g(x) \to g^*(x)$ for all $x \in \Sigma_A^+ \setminus \{\bar 0\}$.

Now we prove that $g^* = 0$ on $\Sigma_A^+ \setminus \{\bar 0\}$. It is enough to show $\mu(|g^*|) = 0$. By Lemma 6.1, $\{\mu(|\mathcal{L}_\psi^n g|)\}$ is decreasing. So $c = \lim_{n\to\infty} \mu(|\mathcal{L}_\psi^n g|)$ exists. By taking subsequences, we obtain that $c = \mu(|\mathcal{L}_\psi^j g^*|)$ for all limit points g^* of $\{\mathcal{L}_\psi^n g\}$ and for all $j \ge 0$. If $c \ne 0$, then by continuity of g^*, we can find $x \in \Sigma_A^+\backslash\{\bar 0\}$, and l-words u and v such that $g^*(ux) > 0$ and $g^*(vx) < 0$. By Lemma 6.1, we have $\mu(|\mathcal{L}_\psi^l g^*|) < \mu(|g^*|)$, a contradiction. So we get $c = 0$. Hence, $\mathcal{L}_\psi^{n_i} g(x) \to 0$ in $L^1(\mu)$. Since this is true for any subsequence, we get $\mathcal{L}_\psi^n g(x) \to 0$ in $L^1(\mu)$. The arguments in the previous paragraph imply that the convergence is uniform on Q_k for any k.

For the case $x = \bar 0$ and $\lambda > 1$, we first note that as $j \to \infty$, $\mathcal{L}_\psi^j g(s\bar 0) \to 0$ for any $s \ne 0$ by the above argumentsand therefore $(\overline{\mathcal{L}_\psi^j g})(\bar 0) \to 0$ by (6.1). Also, by Lemma 5.2, $e^{\psi(\bar 0)} = \lambda^{-1} < 0$ and therefore $e^{S_n\psi(0^n\bar 0)} = \lambda^{-n} \to 0$ as $n \to \infty$. So by Lemma 6.2, $\mathcal{L}_\psi^n g(\bar 0) \to 0$ as $n \to \infty$.

It remains to show that $\lim_{n\to\infty} \big(\mathcal{L}_\psi^n g\big)(\bar 0) = g(\bar 0)$ if $\lambda = 1$. In fact, $\lambda = 1$ implies that $e^{\psi(\bar 0)} = 1$ by Lemma 5.2, and hence $e^{\psi(s\bar 0)} = 0$ for any $s \ne 0$. So $\big(\mathcal{L}_\psi g\big)(\bar 0) = g(\bar 0)$. We get the result. □

Proposition 6.4. *(a) If μ is a probability measure, then for any continuous function g on Σ_A^+,*

$$\lim_{n\to\infty} \mathcal{L}_\psi^n g(x) = \begin{cases} \mu(g) & \text{if } \lambda > 1 \quad \text{or } x \ne \bar 0; \\ g(\bar 0) & \text{if } \lambda = 1 \quad \text{and } x = \bar 0. \end{cases}$$

(b) If μ is an infinite measure, then for any continuous function g on Σ_A^+ with with $\mu(|g|) < \infty$ and $\mu(g) = 0$,

$$\lim_{n\to\infty} \mathcal{L}_\psi^n g(x) = 0 \qquad \forall x \in \Sigma_A^+.$$

The convergence in both cases is in $L^1(\mu)$ and uniform on Q_k for any $k \ge 0$.

Proof. Note that $\mathcal{L}_\psi^n\big(g(x) - \mu(g)\big) = \mathcal{L}_\psi^n g(x) - \mu(g)$. So part (a) can be obtained from Lemma 6.3 by applying the function $g - \mu(g)$.

For part (b), by Lemma 6.3 we only need to show that $g(\bar 0) = 0$. In fact, if $g(\bar 0) > 0$, then by continuity $g(x) > g(\bar 0)/2$ on $\mathcal{O}_k$ for some $k > 0$. Since $\mu(\mathcal{O}_k) = \infty$, we have $\mu(g\chi_{\mathcal{O}_k}) = \infty$. It implies that g is not integrable, a contradiction. □

Proof of Theorem B. By a theorem of Lin ([Li]; see also [A], Theorem 1.3.3), for a nonsingular system (σ, μ), it is exact if and only if $\|\mathcal{L}_\psi^n g\|_1 \to 0$ for any $g \in L^1(\mu)$ with $\mu(g) = 0$, where $\|\cdot\|_1$ denotes the L^1 norm.

Take $g \in L^1(\mu)$ with $\mu(g) = 0$. We need show that for any $\epsilon > 0$, there is $N > 0$ such that for any $n > N$, $\|\mathcal{L}_\psi^n g\|_1 \le \epsilon$.

Since $g \in L^1(\mu)$, we can take $k > 0$ such that $\mu(|g\chi_{\mathcal{O}_{k+1}}|) \le \epsilon/6$. Then take a function g'_ϵ such that $g'_\epsilon|_{\mathcal{O}_{k+1}} = 0$, $g'_\epsilon|_{Q_k}$ continuous, and $\mu(|g'_\epsilon - g\chi_{Q_k}|) \le \epsilon/6$. By triangle inequality we have $\mu(|g'_\epsilon - g|) \le \mu(|g'_\epsilon - g\chi_{Q_k}|) + \mu(|g\chi_{\mathcal{O}_{k+1}}|) \le \epsilon/3$. So we can take a continuous function g_ϵ such that $g_\epsilon(\bar{0}) = 0$, $\mu(|g_\epsilon - g'_\epsilon|) \le \epsilon/3$ and $\mu(g_\epsilon) = 0$. Now we have $\mu(|g - g_\epsilon|) \le 2\epsilon/3$. By Lemma 6.1, for any $n > 0$, $\mu(|\mathcal{L}^n_\psi g - \mathcal{L}^n_\psi g_\epsilon|) \le \mu(\mathcal{L}^n_\psi |g - g_\epsilon|) \le 2\epsilon/3$.

By Proposition 6.4, $\mathcal{L}^n_\psi g_\epsilon \to 0$ in $L^1(\mu)$ as $n \to \infty$. So there exists $N > 0$ such that for all $n > N$, $\mu(|\mathcal{L}^n_\psi g_\epsilon|) \le \epsilon/3$. Hence we get

$$\mu(|\mathcal{L}^n_\psi g|) \le \mu(|\mathcal{L}^n_\psi g - \mathcal{L}^n_\psi g_\epsilon|) + \mu(|\mathcal{L}^n_\psi g_\epsilon|) \le \epsilon.$$

This is what we need. □

7. Gibbs properties: Proof of Theorem C

Lemma 7.1. *There is an increasing sequence of positive numbers $\{\bar{C}_n\}$ such that $\lim_{n\to\infty} \frac{1}{n} \log \bar{C}_n = 0$ and $e^{S_n\varphi(0^n x)} \ge \bar{C}_n^{-1}$ for any $x \in \Sigma_A^+$.*

Proof. Recall that $\mathrm{var}_i(\varphi)$ is defined in (1.3). Take $\bar{C}_n = e^{\sum_{i=0}^{n-1} \mathrm{var}_i(\varphi)}$. Since φ is continuous, $\lim_{n\to\infty} \mathrm{var}_n(\varphi) \to 0$. Hence $\lim_{n\to\infty} \frac{1}{n} \log \bar{C}_n = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathrm{var}_i(\varphi) \to 0$.

Since $0^i x \in \mathcal{O}_i$ and $\varphi(\bar{0}) = 0$, $\varphi(0^i x) \ge -\mathrm{var}_i(\varphi)$. So we get

$$e^{S_n\varphi(0^n x)} \ge e^{-\sum_{i=0}^{n-1} \mathrm{var}_i(\varphi)} = \bar{C}_n^{-1}$$

for any $x \in \Sigma_A^+$. □

Lemma 7.2. *There is a sequence of positive numbers $\{C_n\}$ with $\lim_{n\to\infty} \frac{1}{n} \log C_n = 0$ such that for any $x \in P_k$, n-word w with $\mu\mathcal{R}_w < \infty$,*

$$C_n^{-1} \le \frac{\mu\mathcal{R}_w}{\exp\{-n\log\lambda + S_n\varphi(wx)\}} \le C_n.$$

Proof. By Lemma 5.1 and the definition of ψ, for any n-word $w = uv$, where u is an n_0-word, $n_0 \le n$, we have

$$\mu\mathcal{R}_w = \int_{\mathcal{R}_v} e^{S_{n_0}\psi(wz)} d\mu(z) = \frac{1}{\lambda^{n_0}} \int_{\mathcal{R}_v} e^{S_{n_0}\varphi(wz)} h(wz) d\nu(z). \tag{7.1}$$

First, we consider the case $w = 0^n$. If $\mu\mathcal{O}_n < \infty$, then by Corollary A.2 and Lemma 4.3, $\mu\mathcal{O}_n \le \lambda^{-n} C_{\mu,0} n^{-\beta} \le \lambda^{-n} C_{\mu,0}$ if $h^*(\bar{1}) < 0$, i.e. $\lambda > 1$, and $\mu\mathcal{O}_n \le 1$ if $h^*(\bar{1}) = 0$, i.e. $\lambda = 1$. We may assume $C_{\mu,0} \ge 1$ and then get

$$\mu\mathcal{R}_w = \mu\mathcal{O}_n \le \lambda^{-n} C_{\mu,0} \le \lambda^{-n} C_{\mu,0} \bar{C}_n e^{S_n\varphi(0^n x)}. \tag{7.2}$$

On the other hand, by Lemma 4.4, $e^{S_n\varphi(0^n x)} \le B_\varphi$ for any x. So applying (7.1) with $n_0 = n$ and Lemma 7.1, we get

$$\mu\mathcal{R}_w = \mu\mathcal{O}_n \ge \lambda^{-n}\bar{C}_n^{-1}h_- \ge \lambda^{-n}\bar{C}_n^{-1}h_- B_\varphi^{-1}e^{S_n\varphi(0^n x)},$$

where $h_- = \min\{h(x) :\ x \in \Sigma_A^+\}$. By Lemma 4.1, $h_- > 0$. So we can take

$$C_n = \bar{C}_n \max\{C_{\mu,0}, h_-^{-1}B_\varphi\}. \tag{7.3}$$

For the case $w \ne 0^n$, we may assume $w = u0^{n_1}$, where u is an n_0-word whose last symbol is not equal to 0, and $n = n_0 + n_1$. By Proposition 3.1 and (1.2), for any z,

$$e^{S_{n_0}\varphi(wz)}h(wz) \le e^{J_\varphi K_0^\alpha d(0^{n_1}x, 0^{n_1}z)^\theta}e^{S_{n_0}\varphi(wx)}h(wx) \le e^{J_\varphi C_\gamma K_0^\alpha}e^{S_{n_0}\varphi(wx)}h(wx).$$

Hence, using (7.1) with $v = 0^{n_1}$, we get

$$\mu\mathcal{R}_w \le \lambda^{-n_0}e^{J_\varphi C_\gamma K_0^\alpha}e^{S_{n_0}\varphi(wx)}h(wx)\nu\mathcal{O}_{n_1}.$$

Similarly to (7.2) we get $\nu\mathcal{O}_{n_1} \le \lambda^{-n_1}C_{\nu,0}\bar{C}_n e^{S_{n_1}\varphi(0^{n_1}x)}$. Note that

$$S_n\varphi(wx) = S_{n_0}\varphi(wx)S_{n_1}\varphi(0^{n_1}x).$$

By (3.3), $h(wx) \le e^{J^* n_0^{1-\theta(1+\gamma)+\alpha}} \le e^{J^* n^{1-\theta(1+\gamma)+\alpha}}$. So we have

$$\mu\mathcal{R}_w \le \lambda^{-n}e^{J_\varphi C_\gamma K_0^\alpha}C_{\nu,0}e^{J^* n^{1-\theta(1+\gamma)+\alpha}}\bar{C}_n e^{S_n\varphi(wx)}. \tag{7.4}$$

Similarly, we have

$$\mu\mathcal{R}_w \ge \lambda^{-n_0}e^{-J_\varphi C_\gamma K_0^\alpha}e^{S_{n_0}\varphi(wx)}h(wx)\nu\mathcal{O}_{n_1}$$

and $\nu\mathcal{O}_{n_1} \ge \lambda^{-n_1}\bar{C}_n^{-1}B_\varphi^{-1}e^{S_n\varphi(0^n x)}$. So

$$\mu\mathcal{R}_w \ge \lambda^{-n}e^{-J_\varphi C_\gamma K_0^\alpha}\bar{C}_n^{-1}h_- B_\varphi^{-1}e^{S_n\varphi(wx)}. \tag{7.5}$$

By (7.4) and (7.5), in this case we can take

$$C_n = \bar{C}_n e^{J_\varphi C_\gamma K_0^\alpha}\max\{C_{\nu,0}e^{J^* n^{1-\theta(1+\gamma)+\alpha}}, h_-^{-1}B_\varphi\}. \tag{7.6}$$

Now we take C_n as the larger one in (7.3) and (7.6). Clearly $\{C_n\}$ is subexponential and the inequalities of the lemma are satisfied. $\square$

Lemma 7.3. *There exists a constant $A_\varphi > 0$ such that for any n-word $w = w_0w_1\cdots w_{n-1}$ with $w_{n-1} \neq 0$,*

$$A_\varphi^{-1}h(wx) \leq \frac{\mu\mathcal{R}_w}{\exp\{-n\log\lambda + S_n\varphi(wx)\}} \leq A_\varphi h(wx).$$

Proof. We may assume that $d(x,z)^\theta \leq C_\gamma$ for any $x, z \in \Sigma_A^+$. Since $h \in \overline{\mathcal{G}}_{J_\varphi}$, by Proposition 3.1 ii), we have $e^{-J_\varphi C_\gamma K_0^\alpha} \leq e^{S_n\varphi(wz)}h(wz)/e^{S_n\varphi(wx)}h(wx) \leq e^{J_\varphi C_\gamma K_0^\alpha}$. So by (7.1)

$$\mu\mathcal{R}_w \leq \frac{1}{\lambda^n}\int e^{J_\varphi C_\gamma K_0^\alpha}e^{S_n\varphi(wx)}h(wx)d\nu(z) \leq \frac{1}{\lambda^n}e^{J_\varphi C_\gamma K_0^\alpha}e^{S_n\varphi(wx)}h(wx).$$

On the other hand, if $wx \in \Sigma_A^+$, then

$$\mu\mathcal{R}_w \geq \frac{1}{\lambda^n}\int e^{-J_\varphi C_\gamma K_0^\alpha}e^{S_n\varphi(wx)}h(wx)d\nu(z) \geq \frac{1}{\lambda^n}e^{-J_\varphi C_\gamma K_0^\alpha}e^{S_n\varphi(wx)}h(wx).$$

So we can take $A_\varphi \geq e^{J_\varphi C_\gamma K_0^\alpha}$. □

Note that by [W2] Theorem 9.6, the topological pressure of σ for φ is given by

$$P(\sigma,\varphi) = \limsup_{n\to\infty}\frac{1}{n}\log\sum_{w\in\xi_n}\inf_{wx\in\mathcal{R}_w}e^{S_n\varphi(wx)}. \tag{7.7}$$

Lemma 7.4. *A probability invariant measure μ is an equilibrium state for a continuous function φ whenever it is a weak Gibbs measure for φ, and the constant P in the definition of weak Gibbs measure is equal to the topological pressure $P(\sigma,\varphi)$ for φ.*

Proof. Since μ is a weak Gibbs measure, by (1.9) we have

$$C_n^{-1}e^{nP}\mu\mathcal{R}_{x_0x_1\cdots x_{n-1}} \leq e^{S_n\varphi(x)} \leq C_ne^{nP}\mu\mathcal{R}_{x_0x_1\cdots x_{n-1}}. \tag{7.8}$$

Hence,

$$P - \frac{1}{n}\log C_n \leq \frac{1}{n}S_n\varphi(x) - \frac{1}{n}\log\mu(\mathcal{R}_{x_0x_1\cdots x_{n-1}}) \leq P + \frac{1}{n}\log C_n.$$

Note that $\mathcal{R}_{x_0x_1\cdots x_{n-1}}$ is the element of ξ_n containing x. Let $n \to \infty$, by the Birkhoff Ergodic Theorem, the Shannon-McMillan-Breiman Theorem, and the fact $(1/n)\log C_n \to 0$, we have

$$P = \int\varphi d\mu + h_\mu(\sigma).$$

We show that P is equal to the topological pressure of φ. We replace x by wx in (7.8), where w is an n word, and then use the fact $\sum_{w\in\xi_n}\mu(\mathcal{R}_w) = 1$ to get

$$C_n^{-1}e^{nP} \leq \sum_{\mathcal{R}_w\in\xi_n}\inf_{wx\in\mathcal{R}_w}e^{S_n\varphi(wx)} \leq C_ne^{nP}.$$

By (7.7),

$$P(\sigma,\varphi) = \limsup_{n\to\infty}\frac{1}{n}\log\sum_{w\in\xi_n}\inf_{wx\in\mathcal{R}_w}e^{S_n\varphi(wx)} = P. \qquad \square$$

The result in the lemma is also obtained by M. Yuri (see e.g. [Yu1]) in a slightly different setting.

Lemma 7.5. *Any probability or σ-finite measure μ' that satisfies (1.10) with a function $p'(x,n)$ satisfying a) and b) in Theorem C coincides with μ up to a constant coefficient.*

Proof. Assume μ' satisfies (1.10) with constant P' and function $p'(x,n)$.

Similar arguments as in the proof for Lemma 7.4 show that $P' = P(\sigma,\varphi) = P$.

Fix $k > 0$. Denote $B_k = \sup_{x\in Q_k\cap\sigma^{-n+1}Q_0} p(x,n)$. Let $w = w_0\cdots w_{n-1}$ be an n-word, $n \ge k$, with $\mathcal{R}_w \subset Q_k$.

If $w_{n-1} \ne 0$, then by part b), $p'(wx,n) \le B'_k$ for some $B'_k > 0$. So by (1.10),

$$\mu'\mathcal{R}_w \le B'_k e^{-nP+S_n\varphi(wx)} \le B'_k B_k \mu\mathcal{R}_w. \tag{7.9}$$

If $w_{n-1} = 0$, then we can always find a sequence of words $\{u^{(i)}\}_{i=1}^\infty$ whose last symbols are nonzero such that $\mathcal{R}_w = \{w\bar{0}\} \cup \bigcup_{i=1}^\infty \mathcal{R}_{wu^{(i)}}$. Since μ is an invariant measure, $\mu\{w\bar{0}\} = 0$ for any $w \ne \bar{0}$ because $\bar{0} \in \sigma^{-n}\bar{0}$. By (7.9) we know that

$$\mu'\mathcal{R}_w = \sum_{i=1}^\infty \mu'\mathcal{R}_{wu^{(i)}} \le B'_k B_k \sum_{i=1}^\infty \mu\mathcal{R}_{wu^{(i)}} = B'_k B_k \mu\mathcal{R}_w.$$

Since this is true for all cylinders in Q_k, by taking a limit we know that $\mu'(E) \le B'_k B_k^{-1}\mu(E)$ for all Borel set $E \subset Q_k$. It implies that μ' is absolutely continuous with respect to μ on Q_k and therefore on Σ_A^+. By the Radon-Nykodym Theorem we know that $d\mu'/d\mu$ exists. Since both μ' and μ are σ-invariant, $d\mu'/d\mu$ is a σ-invariant function. Since μ is ergodic, $d\mu'/d\mu$ is equal to a constant c μ-almost everywhere. Clearly, $c > 0$. So we get $\mu' = c\mu$. □

Proof of Theorem C. By Lemma 7.2 with wx replaced by x, μ satisfies (1.9) and therefore is a weak Gibbs measure.

Take $p(x,n) = \min\{C_n, A_\varphi \max\{h(x), h(x)^{-1}\}\}$ if $x_{n-1} \ne \bar{0}$ and $p(x,n) = C_n$ otherwise. Clearly (1.10) holds by Lemma 7.2 and 7.3. Also by Lemma 7.2, $p(x,n)$ satisfies a). Since $h(x)$ is bounded on Q_k for each k, $p(x,n)$ also satisfies b).

The uniqueness follows from Lemma 7.5. □

Proof of Corollary C.1. By Lemma 7.2 and 7.3, we know that P in (1.9) and (1.10) is equal to $\log\lambda$. By Lemma 7.4, it is equal to $P(\sigma,\varphi)$. □

8. Equilibriums: Proof of Theorem D

Since by Theorem C and Lemma 7.4 we know that μ is an equilibrium for φ if μ is a probability measure, the main work in this section is to deal with the case that μ is an infinite measure, in particular, the case $h^*(\bar{1}) = 0$.

Lemma 8.1. *Let $\{a_n\}$ be a decreasing sequence of positive numbers with*

$$\sum_{n=1}^\infty \frac{a_n}{n} < \infty. \tag{8.1}$$

Then $a_n \log n \to 0$ *as* $n \to \infty$.

Proof. First we know that there is a subsequence n_i such that

$$a_{n_i} \log(n_i + 1) \to 0 \qquad \text{as } i \to \infty, \tag{8.2}$$

because if otherwise, there would be $M > 0$ and $\epsilon > 0$ such that $a_n > \epsilon / \log(n+1)$ for all $n \geq M$, and therefore the series in (8.1) would diverge.

Denote

$$S_n = \sum_{i=1}^{n} a_i \log\big(1 + \frac{1}{i}\big), \qquad T_n = \sum_{i=2}^{n} (a_{i-1} - a_i) \log i.$$

Since $\log(1+t) < t$ for any $t > 0$, by (8.1) $S = \lim_{n\to\infty} S_n$ exists. Note that

$$S_n = \sum_{i=1}^{n} a_i \big(\log(i+1) - \log i\big) = a_n \log(n+1) + \sum_{i=2}^{n} (a_{i-1} - a_i) \log i = a_n \log(n+1) + T_n.$$

So T_n is bounded and therefore $T = \lim_{n\to\infty} T_n$ exists. By (8.2) we have $S = T$. Hence $a_n \log(n+1) \to 0$. This implies the result. $\square$

Lemma 8.2. *If* $h^*(\bar{1}) \leq 0$*, then for any* σ*-invariant measure* ρ *with* $\sum_{i=1}^{\infty} i^{-1} \rho P_i < \infty$*, for any* $n > 0$*,*

$$\int_{\sigma^{-1}\mathcal{O}_n \setminus \mathcal{O}_{n+1}} \log h(\sigma x) d\rho(x) < \infty.$$

Suppose Assumption A (III′) also holds; then the condition $h^*(\bar{1}) \leq 0$ *can be removed.*

Proof. Since $h^*(\bar{1}) \leq 0$, by Corollary A.1, $h(\sigma x) \leq Cn$ for some $C > 0$ if $x \in P_n$. So

$$\int_{\sigma^{-1}P_i \setminus P_{i+1}} \log h(\sigma x) d\rho(x) \leq \log(Ci) \rho(\sigma^{-1} P_i \setminus P_{i+1}) = \log(Ci)\big(\rho P_i - \rho P_{i+1}\big),$$

and therefore for any $k \geq n$,

$$\begin{aligned} &\sum_{i=n}^{k} \int_{\sigma^{-1}P_i \setminus P_{i+1}} \log h(\sigma x) d\rho(x) \leq \sum_{i=n}^{k} \log(Ci)\big(\rho P_i - \rho P_{i+1}\big) \\ =&\rho P_n \log(Cn) + \sum_{i=n}^{k} \rho P_{i+1} \big[\log(C(i+1)) - \log(Ci)\big] - \rho P_{k+1} \log(C(k+1)) \\ \leq&\rho P_n \log(Cn) + \sum_{i=n}^{k} i^{-1} \rho P_{i+1} - \rho P_{k+1} \log(C(k+1)). \end{aligned}$$

Since $\sum_{i=n}^{\infty} i^{-1} \rho P_{i+1}$ is convergent and $\rho P_{k+1} \log(C(k+1)) \geq 0$ for all $k \geq n$, the sum in the left side of the inequality is bounded. Note that $\sigma^{-1}\mathcal{O}_n \setminus \mathcal{O}_{n+1}$ is the pairwise disjoint union of the sets $\sigma^{-1}P_i \setminus P_{i+1}$, $i \geq n$; we get the result.

If Assumption A (III′) also holds, then by Corollary A.1, we have $h(x) \leq Cn^{\beta'+1}$ and $\log h(x) \leq (\beta + 1) \log C'n$ for some $C, C' > 0$ if $x \in P_n$. Then the same arguments can be applied. $\square$

Lemma 8.3. *If $h^*(\bar{1}) \le 0$, then for any σ-invariant measure ρ with $\sum_{i=1}^{\infty} i^{-1}\rho P_i < \infty$,*

$$\int \big(\log h(x) - \log h(\sigma x)\big) d\rho(x) = 0.$$

Suppose Assumption A (III′) also holds; then the condition $h^*(\bar{1}) \le 0$ can be removed.

Proof. It is obvious if $\rho(\log h) < \infty$ because ρ is an invariant measure. This is the case if $h^*(\bar{1}) < 0$. So we assume $h^*(\bar{1}) = 0$.

Since $\sigma^{-1}Q_{n-1} = Q_n \setminus \sigma^{-1}\mathcal{O}_n = P_n \cup (Q_{n-1} \setminus \sigma^{-1}\mathcal{O}_n)$ and ρ is an invariant measure, we have

$$\int_{P_n} \log h(\sigma x) d\rho(x) + \int_{Q_{n-1}\setminus\sigma^{-1}\mathcal{O}_n} \log h(\sigma x) d\rho(x) = \int_{Q_{n-1}} \log h(x) d\rho(x).$$

Note that Q_{n-1} can be partitioned into $\{Q_{n-1} \setminus \sigma^{-1}\mathcal{O}_n, \sigma^{-1}\mathcal{O}_n \setminus \mathcal{O}_{n+1}\}$. So

$$\begin{aligned}\int_{Q_{n-1}} \log \frac{h(x)}{h(\sigma x)} d\rho(x) &= \int_{Q_{n-1}} \log h(x) d\rho(x) - \int_{Q_{n-1}} \log h(\sigma x) d\rho(x) \\ &= \int_{P_n} \log h(\sigma x) d\rho(x) - \int_{\sigma^{-1}\mathcal{O}_n\setminus\mathcal{O}_{n+1}} \log h(\sigma x) d\rho(x).\end{aligned}$$

Since Σ_A^+ can be partitioned into $\{Q_{n-1}, \mathcal{O}_n\}$, we only need to prove

$$\int_{\mathcal{O}_n} \log \frac{h(x)}{h(\sigma x)} d\rho(x) = \int_{\sigma^{-1}\mathcal{O}_n\setminus\mathcal{O}_{n+1}} \log h(\sigma x) d\rho(x) - \int_{P_n} \log h(\sigma x) d\rho(x). \tag{8.3}$$

Since $\sigma^{-1}P_i = P_{i+1} \cup (\sigma^{-1}P_i \setminus P_{i+1})$,

$$\int_{P_{i+1}} \log h(\sigma x) d\rho(x) + \int_{\sigma^{-1}P_i\setminus P_{i+1}} \log h(\sigma x) d\rho(x) = \int_{P_i} \log h(x) d\rho(x).$$

Therefore if we denote $\mathcal{O}_{n,k} = \bigcup_{i=n}^{n+k} P_i$ for $k > n$, then

$$\begin{aligned}&\int_{\mathcal{O}_{n,k}} \log \frac{h(x)}{h(\sigma x)} d\rho(x) = \int_{P_{n+k}} \log h(x) d\rho(x) \\ &\qquad + \sum_{i=n}^{n+k-1} \Big[\int_{P_i} \log h(x) d\rho(x) - \int_{P_{i+1}} \log h(\sigma x) d\rho(x)\Big] - \int_{P_n} \log h(\sigma x) d\rho(x) \\ &= \int_{P_{n+k}} \log h(x) d\rho(x) + \sum_{i=n}^{n+k-1} \int_{\sigma^{-1}P_i\setminus P_{i+1}} \log h(\sigma x) d\rho(x) - \int_{P_n} \log h(\sigma x) d\rho(x).\end{aligned}$$

Let $k \to \infty$; we get that this equality implies (8.3). To see this, we first note that by Lemma 8.2 the integral

$$\sum_{i=n}^{\infty} \int_{\sigma^{-1}P_i \setminus P_{i+1}} \log h(\sigma x) d\rho(x) = \int_{\sigma^{-1}\mathcal{O}_n \setminus \mathcal{O}_{n+1}} \log h(\sigma x) d\rho(x)$$

converges. Also, by Corollary A.1, we know that there is $C > 0$ such that for $x \in P_i$, either $h(\sigma x) \le Ci$ if $h^*(\bar{1}) = 0$, or $h(\sigma x) \le Ci^{\beta'+1}$ if Assumption (III′) holds. Using Lemma 8.1 with $a_i = \rho P_i$ we have that if $k \to \infty$, then $\int_{P_{n+k}} \log h(x) d\rho(x) \le \rho P_{n+k} \log C(n+k)^{\max\{1,\beta'+1\}} \to 0$. This completes the proof of the lemma. $\square$

Lemma 8.4. *For any sequence $\{a_n\}$ with $a_n \ge \dfrac{\beta+1}{n+1} - \dfrac{C_\delta}{(n+1)^{1+\delta}}$ and $a_n \to 0$, where δ, C_δ and β are given in Assumption A (III) with $\beta > 0$,*

$$\sum_{k=n}^{\infty} a_k k e^{-\sum_{i=n}^{k} a_i} < \infty \qquad \forall n > 0. \tag{8.4}$$

Proof. By adding the first $n-1$ terms and multiplying by $e^{-\sum_{i=1}^{n-1} a_i}$, we know that if the result is true for some $n > 0$, then it is true for $n = 1$ and therefore for any $n > 0$.

Let $b_n = \dfrac{\beta+1}{n+1} - \dfrac{C_\delta}{(n+1)^{1+\delta}}$.

By the same arguments as in the proof of Lemma 4.4, we know that there exist $B(n) \ge B'(n) > 0$ with $\lim_{n\to\infty} B(n) = \lim_{n\to\infty} B'(n) = 1$ such that for all $k \ge n$,

$$B'(n)\Big(\frac{n}{k+1}\Big)^{\beta+1} \le e^{-\sum_{i=n}^{k} b_i} \le B(n)\Big(\frac{n}{k+1}\Big)^{\beta+1}.$$

Hence, we have

$$\sum_{k=n}^{\infty} b_k k e^{-\sum_{i=n}^{k} b_i} \le \sum_{k=n}^{\infty} \frac{\beta+1}{k+1} k \cdot B(n)\Big(\frac{n}{k+1}\Big)^{\beta+1} < \infty. \tag{8.5}$$

Take $\epsilon > 0$ small such that $\dfrac{\beta+1-\epsilon}{\beta} > 1$. Then take $N > 0$ such that for any $n \ge N$, $b_n \ge \dfrac{\beta+1-\epsilon}{n}$ and $B'(n)\dfrac{\beta+1-\epsilon}{\beta}\Big(\dfrac{n}{n+1}\Big)^{\beta} > 1$. Note that $\sum_{k=n}^{\infty} \dfrac{1}{(k+1)^{\beta+1}} > \int_{n+1}^{\infty} \dfrac{1}{t^{\beta+1}} dt = \dfrac{1}{\beta(n+1)^{\beta}}$. We have that for any $n \ge N$,

$$\begin{aligned}
&\sum_{k=n}^{\infty} b_k k e^{-\sum_{i=n}^{k} b_i} \ge \sum_{k=n}^{\infty} \frac{\beta+1-\epsilon}{k} k \cdot B'(n)\Big(\frac{n}{k+1}\Big)^{\beta+1} \\
\ge &(\beta+1-\epsilon)B'(n)\frac{n^{\beta+1}}{\beta(n+1)^{\beta}} \ge \frac{\beta+1-\epsilon}{\beta} B'(n)\frac{n^{\beta}}{(n+1)^{\beta}} \cdot n > n.
\end{aligned} \tag{8.6}$$

Let

$$c_k^{(j)} = \begin{cases} a_k & \text{if } k < j; \\ b_k & \text{if } k \ge j. \end{cases}$$

Then let

$$S(j) = \sum_{k=n}^{\infty} c_k^{(j)} k e^{-\sum_{i=n}^{k} c_k^{(j)}}.$$

$S(n)$ is the left side of (8.5) and $S(\infty)$ is the left side of (8.4). So by (8.5) we only need to prove that $S(j)$ decreases with j.

Let $T_j(t)$ be the function obtained from $S(j)$ by replacing $c_j^{(j)}$ by t. So we have $T_j(b_j) = S(j)$ and $T_j(a_j) = S(j+1)$. Since $a_j \ge b_j$, we only need to prove that for any $t \ge b_k$, $T_j'(t) < 0$. In fact,

$$T_j'(t) = je^{-t-\sum_{i=n}^{j-1} c_i^{(j)}} - tje^{-t-\sum_{i=n}^{j-1} c_i^{(j)}} - \sum_{k=j+1}^{\infty} c_k^{(j)} k e^{-(t-c_j^{(j)})-\sum_{i=n}^{k} c_i^{(j)}}.$$

Hence by definition of $c_k^{(j)}$ and (8.6), we get

$$e^{t+\sum_{i=n}^{j-1} a_i} T_j'(t) = j - tj - \sum_{k=j+1}^{\infty} b_k k e^{-\sum_{i=j+1}^{k} b_i} < j - tj - (j+1) < 0.$$

It implies $T_j'(t) < 0$. □

Lemma 8.5. *If $h^*(\bar{1}) \le 0$, then $\mu(|\varphi|) < \infty$.*

Proof. If μ is finite, then the result is trivial. So we only need to consider the case that μ is infinite. In this case, $h^*(\bar{1}) = 0$ and $\lambda = 1$.

By Assumption A(III), $\varphi \le 0$ on $\mathcal{O}_{K_1}$. Since $\mu Q_n \le \infty$ for any $n \ge K_1$ and φ is continuous, $\mu(|\varphi|) < \infty$ if and only if $\mu(-\varphi\chi_{\mathcal{O}_n}) < \infty$.

Denote $a_i = \max\{|\varphi(y)| : y \in P_i\}$.

The same arguments as for the proof of Proposition 3.1 give that for any $k \ge n$,

$$\Big|-\sum_{i=n}^{k} a_i - S_{k-n+1}\varphi(0^{k-n+1}x)\Big| \le J_\varphi(\operatorname{diam} P_n)^\theta \le C_\gamma J_\varphi K_0^\alpha \qquad \forall x \in P_n.$$

Since $h^*(\bar{1}) = 0$, by Corollary A.1, we can take n large enough such that $h(x) \le 2k\hat{h}(\bar{0})/\beta$ for any $x \in P_k$ and $k \ge n$. Hence by (4.5) and the definition of ψ, we get

$$\begin{aligned} \mu P_k = &\int_{P_n} e^{S_{k-n+1}\varphi(0^{k-n+1}x)} \frac{h(0^{k-n+1}x)}{h(x)} d\mu(x) \\ \le &e^{-\sum_{i=n}^{k} a_i + C_\gamma J_\varphi K_0^\alpha} \cdot \frac{2k\hat{h}(\bar{0})}{\beta \min\{h(x) : x \in P_n\}} \cdot \mu P_n \le Cke^{-\sum_{i=n}^{k} a_i} \mu P_n, \end{aligned}$$

for some C independent of k. So we have

$$\int_{\mathcal{O}_n} (-\varphi) d\mu = \sum_{k=n}^{\infty} \int_{P_k} (-\varphi) d\mu \le C\, \mu P_n \sum_{k=n}^{\infty} a_k k e^{-\sum_{i=n}^{k} a_i}.$$

By Lemma 8.4, the series on the right side converges if n is large enough. This is what we need. □

Recall that the definition of entropy for an infinite measure is given in (1.11).

Lemma 8.6 (Rohilin's formula). *If μ is an infinite measure, then*

$$h_\mu(\sigma) = -\int \psi d\mu < \infty.$$

Moreover, if $h^*(\bar{1}) = 0$, then

$$h_\mu(\sigma) = -\int \varphi d\mu.$$

Proof. Note that $\beta \le 0$ implies $\lambda > 1$ by Theorem A, and therefore $h^*(\bar{1}) < 0$ by Lemma 4.3. So we have that $h^*(\bar{1}) = 0$ implies $\beta > 0$ and $\lambda = 1$. By Corollary A.2, μP_i is of the order $i^{-\beta}$, and therefore $\sum_{i=1}^{\infty} i^{-1}\mu P_i < \infty$. Hence we can use Lemma 8.3, the definition of ψ, and the fact that $\lambda = 1$ to get $\int \varphi d\mu = \int \psi d\mu$. By Lemma 8.5, the integrals are finite. So we only need to prove the first equality.

The other possibility for μ being an infinite measure is $h^*(\bar{1}) > 1$. In this case the integrals are finite as well because by Lemma 5.3, $|\psi| = -\psi$ is at most of the order $i^{-(\beta+1)}$ with $\beta > 0$ on P_i.

Now we prove $h_\mu(\sigma) = -\int \psi d\mu$.

Take $\Gamma = P_0$ in (1.11). Denote by $\tilde{\sigma}$ and $\tilde{\psi}$ the first return map of σ with respect to P_0 and the corresponding potential, respectively, that is, $\tilde{\sigma}x = \sigma^{n(x)}x$ and $\tilde{\psi}(x) = S_{n(x)}\psi(x)$, where $n(x)$ is the smallest positive integer such that $s^{n(x)}x \in P_0$. Denote by $\tilde{\mu}$ the conditional measure of μ restricted to P_0. Then we define the Perron-Frobenius Operator $\tilde{\mathcal{L}}_{\tilde{\psi}}$ as in (1.4). Using these facts and Lemma 5.1 we can get

$$\int_{P_0} \tilde{\mathcal{L}}_{\tilde{\psi}}\tilde{\psi}(x)d\mu(x) = \int_{\Sigma_A^+} \psi(x)d\mu(x).$$

It means that $\tilde{\mathcal{L}}_{\tilde{\psi}}\tilde{\psi}$ is integrable with respect to μ and therefore with respect to $\tilde{\mu}$. Hence we can get that

$$\int_{P_0} \tilde{\psi}(x)d\tilde{\mu}(x) = \int_{P_0} \tilde{\mathcal{L}}_{\tilde{\psi}}\tilde{\psi}(x)d\tilde{\mu}(x) = \frac{1}{\mu P_0}\int_{P_0} \tilde{\mathcal{L}}_{\tilde{\psi}}\tilde{\psi}(x)d\mu(x). \tag{8.7}$$

Now we calculate $h_{\tilde{\mu}}(\tilde{\sigma})$. Take a partition $\tilde{\xi}$ of P_0 into

$$\{\mathcal{R}_{s0^n} \setminus \mathcal{R}_{s0^{n+1}} : s \neq 0, n = 0, 1, \cdots\}.$$

Clearly, $\tilde{\xi}^- = \bigvee_{i=0}^{\infty} \tilde{\sigma}^{-i}\tilde{\xi}$ is a partition into single points. So $\tilde{\xi}$ is a generator and

$$h_{\tilde{\mu}}(\tilde{\sigma}) = h_{\tilde{\mu}}(\tilde{\sigma}, \tilde{\xi}) = H_{\tilde{\mu}}(\tilde{\xi}|\tilde{\sigma}^{-1}\tilde{\xi}^-) = \int_{P_0} I_{\tilde{\mu}}(\tilde{\xi}|\tilde{\sigma}^{-1}\tilde{\xi}^-)d\tilde{\mu}, \tag{8.8}$$

where $I_{\tilde{\mu}}(\tilde{\xi}|\tilde{\sigma}^{-1}\tilde{\xi}^-)$ is the conditional information of $\tilde{\xi}$ given $\tilde{\sigma}^{-1}\tilde{\xi}^-$ (see [R] or [Ke]). Since the smallest σ-algebra containing $\tilde{\xi}^-$ is the Borel algebra $\mathcal{B}$ over P_0, we have

$$I_{\tilde{\mu}}(\tilde{\xi}|\tilde{\sigma}^{-1}\tilde{\xi}^-) = -\log E_{\tilde{\mu}}(\chi_{\tilde{\xi}(\cdot)}|\tilde{\sigma}^{-1}\mathcal{B}), \tag{8.9}$$

where $E_{\tilde\mu}(\chi_{\tilde\xi(\cdot)}|\tilde\sigma^{-1}\mathcal{B})$ is the conditional expectation of the characteristic function $\chi_{\tilde\xi(\cdot)}$ and $\tilde\xi(x)$ is the element of $\tilde\xi$ that contains x. Note that $\tilde{\mathcal{L}}_{\tilde\psi}$ is the dual operator of the operator $\tilde T$ defined by $\tilde T g(x) = g(\tilde\sigma x)$, and $\tilde\mu$ is a $\tilde\sigma$-invariant measure. So for any $g \in L^1(\tilde\mu)$ and any Borel set $E \subset P_0$, we have

$$\int_{\tilde\sigma^{-1}E} g d\tilde\mu = \int \chi_E(\tilde\sigma x) g(x) d\tilde\mu(x) = \int \chi_E(x)\tilde{\mathcal{L}}_{\tilde\psi} g(x) d\tilde\mu(x)$$
$$= \int_E \tilde{\mathcal{L}}_{\tilde\psi} g(x) d\tilde\mu(x) = \int_{\tilde\sigma^{-1}E} \tilde{\mathcal{L}}_{\tilde\psi} g(\tilde\sigma x) d\tilde\mu(x) = \int_{\tilde\sigma^{-1}E} \sum_{\tilde\sigma y=\tilde\sigma x} e^{\tilde\psi(y)} g(y) d\tilde\mu(x).$$

Since $\tilde{\mathcal{L}}_{\tilde\psi} g \circ \tilde\sigma$ is $\tilde\sigma^{-1}\mathcal{B}$-measurable, we know $E_{\tilde\mu}(g|\tilde\sigma^{-1}\mathcal{B})(x) = \sum\limits_{\tilde\sigma y=\tilde\sigma x} e^{\tilde\psi(y)} g(y)$. Hence,

$$E_{\tilde\mu}(\chi_{\tilde\xi(x)}|\tilde\sigma^{-1}\mathcal{B})(x) = \sum_{\tilde\sigma y=\tilde\sigma x} e^{\tilde\psi(y)}\chi_{\tilde\xi(x)}(y) = e^{\tilde\psi(x)}. \tag{8.10}$$

By (8.8)-(8.10),

$$h_{\tilde\mu}(\tilde\sigma) = -\int_{P_0} \tilde\psi(x) d\tilde\mu(x).$$

By (1.11) and (8.7), we get what we need. □

Lemma 8.7. *If $h^*(\bar 1) > 0$, then $-\int \varphi d\mu = \infty$ and $-\int \psi d\mu < \infty$.*

Proof. By Assumption A(III) we know that $-\varphi$ is bounded below by a function of order n^{-1} over P_n. Also by Corollary A.2, μP_n decreases to a nonzero constant. So we have $-\int \varphi d\mu = \infty$.

By Lemma 5.3 iii), $-\psi$ is of order $k^{-(\beta+1)}$ on P_k. So it is easy to see that $-\int_{\mathcal{O}_n} \psi d\mu < \infty$ for any $n > 0$.

Note that $\psi(x) \to -\infty$ as $x \to s\bar 0$ for any $s \neq 0$. So we need to estimate $\int_{\sigma^{-1}\mathcal{O}_{n-1}\setminus\mathcal{O}_n} \psi d\mu$ as well. Let $R_{s,k} = \{sx : x \in P_{k-1}\}$; then $\sigma^{-1}\mathcal{O}_{n-1} \setminus \mathcal{O}_n = \bigcup\limits_{k=n}^{\infty} \bigcup\limits_{s\neq 0} R_{s,k}$. By a similar method as in the proof of Lemma 5.1, we have

$$\int_{R_{s,k}} \psi(x) d\mu(x) = \int_{P_{k-1}} e^{\psi(sx)}\psi(sx) d\mu(x).$$

So

$$\int_{\sigma^{-1}\mathcal{O}_{n-1}\setminus\mathcal{O}_n} \psi(x) d\mu(x) = \sum_{s\neq 0}\sum_{k=n}^{\infty} \int_{P_{k-1}} e^{\psi(sx)}\psi(sx) d\mu(x).$$

Note that $\varphi(sx)$ and $h(sx)$ are bounded. By the definition of ψ and Corollary A.2 iii),

$$-e^{\psi(sx)}\psi(sx) = \frac{e^{\varphi(sx)}h(sx)}{h(x)}(-\varphi(sx) - \log h(sx) + \log h(x)) \le \frac{C_1}{k^{1+\beta}}(C_2 + \log k^{1+\beta})$$

for some C_1,$C_2 > 0$ independent of k. Since $\beta > 0$, the series $\sum_{k=n}^{\infty} k^{-(1+\beta)} \log k^{1+\beta}$ converges. So we get $-\int_{\sigma^{-1}\mathcal{O}_{n-1}\setminus\mathcal{O}_n} \psi d\mu < \infty$. □

Proof of Theorem D. For the case $h^*(\bar{1}) \leq 0$, if μ is a probability measure, then it is an equilibrium by Theorem C and Lemma 7.4. If it is an infinite measure, then by Theorem A we have $h^*(\bar{1}) = 0$, and therefore by Corollary A.2 we have $\sum_{i=1}^{\infty} i^{-1}\mu P_i < \infty$. Hence by Lemma 8.6 we have Rohlin's formula. Since by Corollary C.1 and Lemma 4.3, $P(\varphi) = \log\lambda = 0$, we get (1.12). On the other hand, if $h^*(\bar{1}) > 0$, then by Lemmas 8.6 and 8.7 we know that Rohilin's formula and therefore (1.12) do not hold for μ.

Consider the case $h^*(\bar{1}) \geq 0$. We know by Lemma 4.3 that $h^*(\bar{1}) \geq 0$ if and only if $P(\varphi) = 0$, and obviously this is true if and only if $\delta_{\bar{0}}$ satisfies (1.12).

Now we prove uniqueness. First we consider the case $h^*(\bar{1}) \leq 0$.

Suppose ρ is a probability ergodic measure. Since the topological entropy of σ is finite and φ is continuous, we have $h_\rho(\sigma) < \infty$ and $P(\varphi) < \infty$. So $\rho(\varphi) < \infty$. By Assumption (III) we have $\varphi(x) < 0$ and $|\varphi(x)| > c/k$ for some $c > 0$ if $x \in P_k$. Hence we have $\sum_{i=1}^{\infty} i^{-1}\rho P_i < \infty$. Then we can apply Lemma 8.3 to get $\int \psi d\rho = \int \varphi d\rho - \log\lambda$. Also $P(\psi) = P(\varphi) - \log\lambda$. Hence, $P(\varphi) = h_\rho(\sigma) + \int \varphi d\rho$ if and only if $P(\psi) = h_\rho(\sigma) + \int \psi d\rho$. That is, ψ and φ have the same equilibriums.

By the same arguments used in the proof of Theorem 10 in [W1], ρ is an equilibrium of ψ if and only if $\mathcal{L}_\psi^*\rho = \rho$, where $\mathcal{L}_\psi^*$ is the dual operator of $\mathcal{L}_\psi$. Note that by Proposition 6.4, for any continuous function g,

$$\mathcal{L}_\psi^n g(x) \to \begin{cases} \mu(g) & \text{if } \lambda > 1 \text{ or } x \neq \bar{0}; \\ g(\bar{0}) & \text{if } \lambda = 1 \text{ and } x = \bar{0}. \end{cases}$$

So if ρ is an ergodic probability measure and equilibrium, then

$$\rho(g) = \big({\mathcal{L}_\psi^*}^n\rho\big)(g) = \rho\big(\mathcal{L}_\psi^n g\big) \to \begin{cases} \rho(\mu(g)) = \mu(g) & \text{if } \lambda > 1 \text{ or } \rho(\{\bar{0}\}) = 0; \\ \rho(g(\bar{0})) = g(\bar{0}) & \text{if } \lambda = 1 \text{ and } \rho(\{\bar{0}\}) = 1. \end{cases}$$

Hence, if $P(\varphi) > 0$, then $\rho = \mu$, and if $P(\varphi) = 0$, then either $\rho = \mu$ or $\rho = \delta_{\bar{0}}$.

Now we consider the case $h^*(\bar{1}) > 0$. If a probability measure ρ is an equilibrium for φ, then by Lemma 4.6 i), there is $\varphi' \geq \varphi$ satisfying Assumptions (I)-(III) with $\varphi'(x) > \varphi(x)$ for some $x \neq \bar{0}$ such that the corresponding ${h'}^*(\bar{1}) > 0$. Hence by Lemma 4.3, we have $\lambda' = 1$ and therefore $P(\varphi') = 0$. Now it follows that

$$P(\varphi') = 0 = P(\varphi) = h_\rho(\sigma) + \int \varphi d\rho < h_\rho(\sigma) + \int \varphi' d\rho.$$

It contradicts the variational principle. So there is no probability equilibrium ρ with $\rho(\Sigma_A^+ \setminus \{\bar{0}\}) > 0$.

Lastly, we assume that Assumption (III′) also holds. If ρ is an infinite measure such that $\rho(\varphi) < \infty$, then the same argument as above gives $\sum_{i=1}^{\infty} i^{-1}\rho P_i < \infty$. Hence we can use Lemma 8.3 to get $\rho(\varphi) = \rho(\psi)$. Then the same arguments as in the case $h^*(\bar{1}) \leq 0$ gives that if ρ satisfies (1.12), then $\rho = \mu$. □

Proof of Corollary D.1. Note that if φ satisfies Assumptions A(I)-(III), then so is $t\varphi$ for all $t > 0$. Since $\varphi(x) \leq 0$ for all x and $t > 0$, $P(t\varphi)$ decreases with t. It is easy to see that if t is large enough, then $P(t\varphi) = 0$.

Let $t_0 = \min\{t : P(t\varphi) = 0\}$. Clearly $t_0 > 0$ since $P(0) > 0$. So if $t < t_0$, then $P(t\varphi) > 0$ and by Theorem D, μ is the unique equilibrium. We get part i).

For $t = t_0$, by Lemma 4.6 i) we know that the corresponding $h^*_{t_0\varphi}(\bar{1}) = 0$. In fact, if it is less than 0, then we can find $t > t_0$ such that $h^*_{t\varphi}(\bar{1}) < 0$, and by Lemma 4.3, we have $\lambda_{t\varphi} = 0$ and therefore $P(t\varphi) = 0$, contradicting the choice of t_0. So by Theorem D, we know that both μ and $\delta_{\bar{0}}$ satisfy (1.12). Whether μ is an equilibrium depends whether μ is finite. By Theorem A, this depends on whether $t_0(\beta + 1) - 1 > 1$. So part ii) is true.

For part iii), we know by Lemma 4.6 ii) that $h^*_{t\varphi}(\bar{1}) > 0$. By Theorem D we know that $\delta_{\bar{0}}$ is an equilibrium for $t\varphi$, and $\mu_{t\varphi}$ is an infinite measure and does not satisfy (1.12). $\square$

Proof of Corollary D.2. By Lemma 6.3, any σ-invariant weak Gibbs measure ρ for φ is an equilibrium for φ. Since μ, and probably $\delta_{\bar{0}}$, are the only equilibriums for φ, and $\delta_{\bar{0}}$ is not a weak Gibbs measure, we must have $\rho = \mu$. $\square$

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