Absolutely Continuous Invariant Measures for Nonuniformly Expanding Maps

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Abstract

For a large class of nonuniformly expanding maps of $\mathbb{R}^m$, with indifferent fixed points and unbounded distortion and non necessarily Markovian, we construct an absolutely continuous invariant measure. We extend to our case techniques previously used for expanding maps on quasi-Hölder spaces. We give general conditions and provide examples to which apply our result.

0 Introduction

The aim of this paper is to treat a class of multidimensional nonsingular transformations with indifferent fixed points which do not enjoy any Markov property. These maps exhibit two major difficulties. First the presence of discontinuities (the boundaries of the domains of local injectivity); second the nonuniformity caused by the presence of the indifferent fixed points. While there are several techniques to handle with the former point (see e.g. [C], [Bu], [S]), there is an essential difficulty for the latter one: unbounded distortion.

It is well known that for a nonuniformly expanding map T on the unit interval with an indifferent fixed point p, unbounded distortion occurs at the fixed point (see for instance the examples treated in [Pi] and [Th]). That is, for any neighborhood U of p, there are points $x \in U$ such that $|(T_1^{-n})'(p)/(T_1^{-n})'(x)|$ is unbounded as n increases, where T_1 denote the restriction of T to some neighborhood of p. However, this only happens at the indifferent fixed point p; if we remove an arbitrarily small neighborhood of p, and we consider the first return

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map with respect to its complement, then the reduced system has bounded distortion. Unfortunately, this is not the case for nonuniformly expanding maps T in higher dimensional spaces with an indifferent fixed point p (we just consider only one indifferent fixed point for convenience). The system has *unbounded distortion* in the following sense: there are uncountably many points z such that for any neighborhood V of z, we can find $\hat{z} \in V$ such that the ratio

$$|\det DT_1^{-n}(z)|/|\det DT_1^{-n}(\hat{z})|$$

is unbounded as $n \to \infty$ (see Example 1 in Section 2). The points with unbounded distortion are not given explicitly by the map. Therefore, the methods for the one dimensional case cannot be applied directly here, since the first return map with respect to the complement of any neighborhood of p still have unbounded distortion.

Distortion estimates play an important role for the existence of absolutely continuous invariant measures in the case of hyperbolic or expanding maps. This is because the bounds of distortion give the bounds of the ratio of the density function. In many works, bounded distortion are either assumed or proved (e.g. [Yo1, Yo2], [ABV], [FJ], [Yu1, Yu2], [BPS]). However, for many systems the density function $h(x)$ may be only an L^1 function and, the ratio $h(x)/h(y)$ may be unbounded as well on close points x and y. Therefore we need some techniques to handle these situations.

This work is an attempt toward this direction: we will prove the existence of absolutely continuous invariant measures for maps with indifferent fixed points in higher dimensions and in presence of unbounded distortion.

Existence of absolutely continuous invariant measures for expanding systems with an indifferent fixed point was proved for one dimensional cases in 1980 ([Pi], [Th]). However, there is no corresponding results for higher dimensional cases, except for some special examples ([Yu2], [H]).

In this paper we are able to cover an open set of maps in the space of expanding systems with an indifferent fixed point p whose local expression is an isometry plus homogeneous terms and higher order terms (see Example 3 and Remark 2.1 thereafter). We could also deal with maps whose differential has at least one eigenvalue greater than one at the indifferent fixed point (see Example 2). Actually our assumptions (1 to 4) are formulated in a general way with the attempt to capture and control the delicate behavior around the indifferent fixed point due to the lack of bounded distortion. We provide in Theorem B and C sufficient conditions to check those assumptions and we successively apply them to examples in Section 2.

What we get here could not be derived easily from other existing results. Since the distortions are unbounded, Young's results [Yo1, Yo2] do not follow directly. Neither do Yuri's techniques [Yu1, Yu2] for the same reasons, even if we model our map to give it a Markov or a finite range structure. Also, the condition

$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log ||DT(T^i(x))^{-1}|| < 0$ in [ABV] cannot be obtained in our case since T may admit a σ-finite absolutely continuous invariant measure. If we study the first return map $\hat{T}$ instead, then $\|D\hat{T}_x(v)\|$ can be arbitrary large for x close to the discontinuity set, and therefore the assumptions on the critical set in [ABV] are not satisfied.

Our construction consists of the following steps. We first replace the transformation T by a first return map $\hat{T}$ with respect to a domain outside a neighborhood of the indifferent fixed point to get a uniformly expanding map with a countable number of discontinuity surfaces. Then we prove a Lasota-Yorke [LY] inequality on the induced system by acting the Perron-Frobenius (PF) operator on the space of "quasi-Hölder" functions to obtain a density function $\hat{h}$. This result is interesting in itself since it extends the work of [S] to piecewise expanding maps with a countable number of branches. The density function $\hat{h}$ defines an absolutely continuous probability measure $\hat{\mu}$ invariant under $\hat{T}$; then we extend $\hat{\mu}$ to an absolutely continuous invariant measure μ for T. The measure μ has finitely many ergodic components, and these could be finite or infinite, depending on the behavior of T near the fixed point. Moreover these maps can be arranged in such a way that the absolutely continuous invariant measure has both finite and infinite components that lie side by side (see Example 1).

The space of quasi-Hölder functions, introduced by Keller [K], developed by Blank [Bl] and successfully applied by Saussol [S] and successively by Buzzi [Bu] (see also [BK]) and Tsujii [Ts] to the multidimensional expanding case, reveals to be very useful to control the oscillations of a function under the iteration of the PF operator across the discontinuities of the map *. Our result shows that it is also useful for unbounded distortion caused by nonuniform expansion, since as we point out in Remark 1.9, the oscillations of test functions are produced under iteration, not only by the propagation of discontinuities, but also by the distortion of the determinant of the map.

The plan of the paper is the following: in Section 1 we state the assumptions and the main theorem (Theorem A) about existence of absolutely continuous invariant measures, and give sufficient conditions (Theorems B and C) to produce a wide class of maps which fit the preceding assumptions. In Section 2 we study carefully some concrete examples. The proofs of Theorem B and C are respectively in Section 3 and 4. Section 5 contains the proof of Theorem A, while the last section, Section 6, deals with the proof of the Lasota-Yorke inequality for the induced system.

*The use of the more standard space of bounded variation functions (in the sense of distribution) allowed as well to get absolute invariant measures for some class of piecewise uniformly expanding maps, see for instance [BG, PGB, Ad, C], but they need a stronger control of the geometrical shape of the discontinuity surfaces.

1 Assumptions and statements of results

Let $M \subset \mathbb{R}^m$ be a compact subset with $\overline{\operatorname{int} M} = M$ and d be the Euclidean distance. Let ν be the Lebesgue measure on M. We assume $\nu M = 1$.

Let $T : M \to M$ be a map satisfying the following assumptions.

For $A \subset M$ and $\varepsilon > 0$, denote $B_\varepsilon(A) = \{x \in \mathbb{R}^m; d(x, A) \le \varepsilon\}$.

Assumption 1. (Piecewise smoothness) *There are finitely many disjoint open sets $U_1, \cdots, U_K$ with $M = \bigcup_{i=1}^K \overline{U_i}$ such that for each i,*

(a) *$T_i := T|_{U_i} : U_i \to M$ is $C^{1+\alpha}$;*

(b) *T_i can be extended to a $C^{1+\alpha}$ map $T_i : \tilde{U}_i \to M$ such that $T_i\tilde{U}_i \supset B_{\varepsilon_1}(T_iU_i)$ for some $\varepsilon_1 > 0$, where $\tilde{U}_i$ is a neighborhood of U_i.*

Assumption 2. (Fixed point) *There is a point $p \in U_1$ such that:*

(a) *$Tp = p$;*

(b) *$T^{-1}\{p\} \notin \partial U_j$ for any j.*

Since $M \subset \mathbb{R}^m$, we may take a coordinate system such that $p = 0$. Hence, we write $|x| = d(x, p)$ if $x \in M$.

For any $x \in U_i$, we define $s(x) = s(x, T)$ as the inverse of the slowest expansion near x by:

$$s(x, T) = \min\big\{s : d(x, y) \le sd(Tx, Ty), y \in U_i, d(x, y) \le \min\{\varepsilon_1, 0.1|x|\}\ \big\}.$$

where the factor 0.1 makes the ball away from the origin, though any other small factor would work as well.

Denote by γ_m the volume of the unit ball in $\mathbb{R}^m$.

Assumption 3. (Expanding Rates) *There exists an open connected region R bounded by a smooth surface with $p \in R$, $\overline{R} \subset TR$, $\overline{TR} \subset U_1$ and with either $\overline{R} \subset TU_j$ or $\overline{R} \cap TU_j = \emptyset$ such that:*

(a) *$\forall x \in M \setminus \{p\}$ we have $0 < s(x) \le 1$, and if $s(x) = 1$ then $x \in R$ and $|Tx| > |x|$;*

(b) *there exist constants $\eta_0 \in (0, 1)$, $\varepsilon_2 > 0$ such that*

$$s^\alpha + \lambda \le \eta_0 < 1,$$

where

$$s := \max\{s(x) : x \in M \backslash R\},$$

$$\lambda = \max\Big\{2 \sup_{\varepsilon_0 \le \varepsilon_2} \sup_{\varepsilon \le \varepsilon_0} \frac{G_U(\varepsilon, \varepsilon_0)}{\varepsilon^\alpha}\varepsilon_0^\alpha,\ \frac{3s\gamma_{m-1}}{(1-s)\gamma_m}\Big\}, \tag{1.1}$$

$$G_U(\varepsilon, \varepsilon_0) = \sup_{x \in M} G_U(x, \varepsilon, \varepsilon_0), \tag{1.2}$$

and

$$G_U(x, \varepsilon, \varepsilon_0) = \sum_{j=1}^{K} \frac{\nu(T_j^{-1} B_\varepsilon(\partial T U_j) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))};$$

(c) *there exists* $N = N_s > 0$ *and* $\varepsilon_3 > 0$ *such that for all* $x \in B_{\varepsilon_3}(TR \backslash R)$,

$$s(T_1^{-N}(x), T_1^N) \leq \frac{s}{5m} \Big(\frac{\lambda(1-s)^m}{2C_p I^2} \Big)^{1/\alpha}$$

for λ *given by (1.1) and where* I *and* C_p *are constants defined below in Assumption 4(c).*

We say that $T : M \to M$ is an *almost expanding piecewise smooth map with an indifferent fixed point* p if it verifies Assumption 1, 2 and 3(a), and $s(p) = 1$.

Remark 1.1. *We first observe that Assumption 1 does not require any Markov property of the partition* $\{U_1, \cdots, U_K\}$. *Moreover, by Assumption 3(a), the map* $T_j : U_j \to T_j(U_j)$ *is noncontracting for each* j, *and therefore it is a local diffeomorphisms. Also, by the assumption, for any* $x \in U_1$, $T_1^{-n} x \to p$, *because the set of limit points of* $\{T_1^{-n} x\}$ *cannot contain any other point but* p.

Remark 1.2. *By Assumption 1(b),* $|detDT(x)|$ *is bounded. This is because the map* $x \to |detDT(x)|$ *is continuous on* $\overline{U}_i$ *for each* i, *and* $\overline{U}_i$ *is compact.*

Remark 1.3. *Assumption 3(b) is the main assumption that requires uniform dilation outside* R *and gives condition on the relations between expanding rates and discontinuity. It is proved in [S] (see the proof of Lemma 2.1) that if the boundary of* U_i *consists of piecewise* C^1 *codimension one embedded compact submanifolds, then* $G_U(\varepsilon, \varepsilon_0) \leq 2Y \frac{\gamma_{m-1}}{\gamma_m} \frac{s\varepsilon}{(1-s)\varepsilon_0} \big(1 + o(1)\big)$, *where* Y *is the maximal number of smooth components of the boundary of all* U_i *that meet in one point. We refer to [S] for more details about the meaning of* $G_U(\varepsilon, \varepsilon_0)$.

Remark 1.4. *We do not require the boundary of* U_i *to be piecewise smooth. In fact, they could be fractals as analyzed in [S]. However, Assumption 3(b) implies* $\nu(\partial U_j) = 0$ *for any* $j = 1, \cdots, K$. †

† In fact, if $\nu(\partial U_j) > 0$ for some j, then we take the set of the *density points*

$$\Delta = \Big\{ x \in M : \lim_{\varepsilon \to 0} \frac{\nu(B_\varepsilon(x) \cap \partial U_j)}{\nu B_\varepsilon(x)} = 1 \Big\}.$$

By the Lebesgue-Vitali Theorem (see, e.g. [SG], Chapter 10), $\nu\Delta = \nu(\partial U_j) > 0$. In particular, $\Delta \neq \emptyset$. Therefore for any $x \in \Delta$, if ε_0 is sufficiently small and $\varepsilon = (1-s)\varepsilon_0$, then

$$G_U(x, \varepsilon, \varepsilon_0) \geq \frac{\nu(T_j^{-1} B_\varepsilon(\partial T U_j) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))} \geq \frac{\nu(\partial U_j \cap B_\varepsilon(x))}{\nu(B_\varepsilon(x))}$$

is sufficiently close to 1, which contradicts the assumption.

Remark 1.5. *We allow that $s(x,T) = 1$ for some x other than p. However we still need some expanding rate inside R. This is given by Assumption 3(c). If $s(T_1^{-N}(x), T_1^N)$ can be arbitrarily small by taking N sufficiently large, then Assumption 3(c) is always true.*

Denote $R_0 = TR\backslash R$. Clearly, $R_0 \subset U_1$ because of the choice of R.

Assumption 4. (Distortions)

(a) *There exists $c > 0$ such that for any $x, y \in TU_j$ with $d(x,y) \le \varepsilon_1$,*

$$\left|\det DT_j^{-1}(x) - \det DT_j^{-1}(y)\right| \le c|\det DT_j^{-1}(x)|d(x,y)^\alpha,$$

where ε_1 is given by Assumption 1(b);

(b) *For any $b > 0$, there exist $J > 0$, $\varepsilon_4 > 0$ such that for any $\varepsilon \in (0, \varepsilon_4]$, we can find $0 < N = N(\varepsilon) \le \infty$ with*

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|} \le 1 + J\varepsilon^\alpha \quad \forall y \in B_\varepsilon(x),\ x \in B_{\varepsilon_4}(R_0),\ n \in (0, N],$$

and

$$\sum_{n=N}^{\infty} \sup_{y \in B_\varepsilon(x)} |\det DT_1^{-n}(y)| \le b\varepsilon^{m+\alpha} \quad \forall x \in B_{\varepsilon_4}(R_0);$$

(c) *There exist constants $I > 1$, $C_p > 0$, $\varepsilon_5 > 0$ such that for any $0 < \varepsilon_0 \le \varepsilon_5$, $n > 0$, there is a finite or countable partition $\xi = \xi_n$ of $B_{\varepsilon_0}(R_0)$ such that $\forall A \in \xi$, $0 < \varepsilon \le \varepsilon_0$, $\mathrm{diam}(A \cap B_{\varepsilon_0}(\partial R_0)) \le 5m\varepsilon_0$,*

$$\frac{\nu\big(B_\varepsilon(\partial R_0) \cap A\big)}{\nu\big(B_{\varepsilon_0}(\partial R_0) \cap A\big)} \le C_p \left(\frac{\varepsilon}{\varepsilon_0}\right)^\alpha, \tag{1.3}$$

whenever $\nu\big(T_1^{-n}(B_{\varepsilon_0}(\partial R_0)) \cap A\big) \ne 0$, and for any $x, y \in A$,

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|} \le I. \tag{1.4}$$

Remark 1.6. *In fact, Assumption 4(a) is a consequence of Assumption 1(a) since $|detDT(x)|$ is bounded from above as we mentioned in Remark 1.2. However we state it here independently due to its importance for our arguments.*

Remark 1.7. *If T_1^{-1} has bounded distortion in $B_{\varepsilon_5}(R_0)$ in the sense that for any $J_0 > 1$, there is $\varepsilon > 0$ such that for any $x, y \in B_{\varepsilon_5}(R_0)$ with $d(x,y) \le \varepsilon$ and for any $n > 0$, $\dfrac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|} \le J_0 d(x,y)^\alpha$, then Assumption 4(b) and (c) are true with $\varepsilon_4 = \varepsilon_5 = \varepsilon_0$.*

Remark 1.8. *Actually, by our proof the condition* $\operatorname{diam}(A\cap B_{\varepsilon_0}(\partial R_0)) \le 5m\varepsilon_0$ *in Assumption 4(c) can be replaced by*

$$\operatorname{diam} T_1^{-n}(A \cap B_{\varepsilon_0}(\partial R_0)) \le s\Big(\frac{\lambda(1-s)^m}{2C_p I^2}\Big)^{1/\alpha}$$

for all $n \ge N_s$*, where* s *and* N_s *are given by Assumption 3(b) and (c) respectively (see (6.4)).*

Remark 1.9. *When we iterate the system, oscillations of the test functions are produced by both discontinuities* ∂U_j *and distortion of* $|\det DT|$*. It is very common for an expanding system in a multidimensional space with an indifferent fixed point to have unbounded distortion near the fixed point, (see Example 1 in the next section). Assumption 4(b) requires that the distortion (namely the ratio* $\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|}$*) is uniformly bounded only up to some* $n = N(\epsilon)$ *whenever the points* x *and* y *are* ϵ*-close and* $|\det DT_1^{-n}(y)|$ *is summable for* $n \ge N(\epsilon)$ *with the sum which is small (of the order of a power of* ϵ*).*

Remark 1.10. *Assumption 4(c) is a usual bounded distortion estimate since it controls uniformly the ratio* $\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|}$ *when the points* x *and* y *are chosen close enough in an element of a partition depending on* n*.*

We are now ready to state our main result.

Theorem A. *Suppose* $T : M \to M$ *satisfies Assumption 1 to 4. Then* T *admits an absolutely continuous invariant measure* μ *with at most finitely many ergodic components* $\mu_1, \cdots, \mu_s$ *that are either finite or* σ*-finite, and the density functions of* μ_i *are bounded on any compact set away from* p*. Hence,*

- μ *is finite if* $\displaystyle\sum_{n=1}^{\infty} \nu(T_1^{-n}R) < \infty$.

Moreover, if for any ball $B_\varepsilon(x)$ *in* M*, there exists* $\tilde N = \tilde N(x,\varepsilon) > 0$ *such that* $T^{\tilde N} B_\varepsilon(x) \supset M$*, then the density function is bounded below by a positive number. Hence*

- μ *is* σ*-finite if* $\displaystyle\sum_{n=1}^{\infty} \nu(T_1^{-n}R) = \infty$.

Remark 1.11. *We will give an example in the next section showing that it is possible for* μ *to have both finite and* σ*-finite ergodic components simultaneously, and both contain the same indifferent fixed point* p *in their supports.*

Since Assumption 4(b) and 4(c) are difficult to verify, we give some sufficient conditions in the next theorems.

The conditions satisfied by the maps studied in the second part of both Theorem B and C are the following:

There are constants $\gamma' > \gamma > 0$, $C_i, C_i' > 0$, $i = 0, 1, 2$, such that in a neighborhood of the indifferent fixed point $p = 0$,

$$|x|\big(1 - C_0'|x|^\gamma + O(|x|^{\gamma'})\big) \leq \ |T_1^{-1}x| \leq |x|\big(1 - C_0|x|^\gamma + O(|x|^{\gamma'})\big), \quad (1.5)$$

$$1 - C_1'|x|^\gamma \leq \|DT_1^{-1}(x)\| \leq 1 - C_1|x|^\gamma, \quad (1.6)$$

$$C_2'|x|^{\gamma-1} \leq \|D^2T_1^{-1}(x)\| \leq C_2|x|^{\gamma-1}. \quad (1.7)$$

Remark 1.12. *If a map T_1 satisfies all of these inequalities, then $\|DT_1(p)\| = \|DT_1(p)^{-1}\| = 1$, and $DT_1(p)$ is an isometry. If $\gamma > 1$, then $D^2T_1(p) = 0$.*

Further, consider the space of C^3 maps that is expanding at every point x in a neighborhood of p except at p itself and whose differential $DT_1(p)$ is an isometry. Then it is easy to see that the maps satisfying (1.5) to (1.7) form a generic set in the C^3 topology.

We would like to point out that the local behaviors given by the inequalities (1.5) to (1.7) allow us to apply the useful Lemmas 3.1 and 3.2 (see below), which permit a good control of the iterates of the Jacobian of the map. This will allow us to check assumptions 4(b) and 4(c) on the examples of the next section.

Theorem B. *Suppose $T : M \to M$ verifies Assumption 1 to 3 and 4(a). Then the Assumption 4(b) is satisfied if one of the following two conditions holds:*

i) There exists a constant $\kappa \in (0, 1)$ such that $|\det DT| \geq \kappa^{-1} > 1$, and a constant $\hat{\alpha} > \alpha$ such that T is $C^{1+\hat{\alpha}}$ in a neighborhood of p. In this case, μ is finite if Assumption 4(c) also holds.

ii) There exists an open region $\tilde{R} \subset R$ containing p with $T_1^{-L}R \subset \tilde{R}$ for some $L > 0$, and constants $\gamma' > \gamma > 0$, $C_0, C_1, C_2 > 0$ such that the second inequalities in (1.5) to (1.7) hold; and there exist constants $\delta, \tau > 0$, $C_\delta, C_\tau > 0$ with

$$\frac{1}{\gamma(1-\alpha)} - \tau < \frac{\delta - 1}{m + \alpha} \quad (1.8)$$

such that for any $x \in R_0$, $n \geq L$,

$$|\det DT_1^{-n}(x)| \leq \frac{C_\delta}{n^\delta}, \qquad \|DT_1^{-n}(x)\| \leq \frac{C_\tau}{n^\tau}. \quad (1.9)$$

In Theorem C, part i) below, we use the partial order $x \prec y$ between two points x and y if $|x_j| < |y_j|$ for every $j = 1, \cdots, m$, where x_j and y_j are the jth component of x and y in $\mathbb{R}^m$ respectively. In part ii) we denote by $E(v_1, \cdots, v_k)$ the subspace spanned by vectors $v_1, \cdots, v_k$, and by $E_x(S)$ the tangent space of a submanifold S at a point $x \in S$. Also, we may use a coordinate system (t, ϕ) near p where $t = |x|$ and $\phi \in \mathbb{S}^{m-1}$, the $m - 1$ dimensional sphere.

Theorem C. *Suppose $T : M \to M$ verifies Assumption 1 to 3 and 4(a). Then the Assumption 4(c) is satisfied if one of the following two conditions holds:*

i) There is a partition of TR into finite or countable number of regions $\{\mathcal{D}_i\}$ such that $\nu(R \setminus \cup_i \mathcal{D}_i) = 0$, $T(\mathcal{D}_i \cap R) = \mathcal{D}_i \cap TR$ and on each $\mathcal{D}_i \cap TR$, with the partial order decribed as above, T satisfies the following: $x \prec y$ implies $|\det DT(x)| < |\det DT(y)|$; if $x \prec y$, then $T_1^{-1}x \prec T_1^{-1}y$; $x \prec Tx$ for any $x \in \mathcal{D}_i \cap R$; and for any $y \in R_0 \cap \mathcal{D}_i$ there is $x \in \partial R \cap \mathcal{D}_i$ such that $x \prec y \prec Tx$.

ii) Suppose T is $C^{1+\gamma}$ and satisfies (1.5) to (1.7) near p. There are two families of cones $\{\mathcal{C}_x\}$ and $\{\mathcal{C}'_x\}$, uniformly continuous in (t, ϕ) in the tangent bundle over the set TR, where $t \geq 0$ and $\phi = \mathbb{S}^{m-1}$ with $(t, \phi) \in TR$, such that (a) $DT_x(\mathcal{C}_x) \subset \mathcal{C}_{Tx}$ and $DT_x(\mathcal{C}'_x) \supset \mathcal{C}'_{Tx}$ $\forall x \in R$; (b) there exists a positive angle θ_0 such that for any $x \in TR$ and $v \in \mathcal{C}_x$ and $v' \in \mathcal{C}'{}_x$, the angle between these two vectors is bounded from below by θ_0; (c) $\exists d > 0$, such that

$$\frac{|\det DT_x|_{E(v,v')}|}{\|DT_x|_{E(v)}\| \cdot \|DT_x|_{E(v')}\|} \leq 1 - d|x|^\gamma \tag{1.10}$$

for any $v, v' \in \mathcal{C}_x$; and (d) $\mathcal{C}_x$ contains the position vector from p to x for all $x \in TR$, $\mathcal{C}'_x$ contains $E_x(\partial B_\varepsilon(R_0))$ for all $x \in \partial(B_\varepsilon(R_0))$, $0 < \varepsilon \leq \varepsilon_5$, and

$$\|DT_x|_{E(\partial(T_1^{-n}R))}\| \leq \frac{|Tx|^{1/(1-\theta)}}{|x|^{1/(1-\theta)}} \quad \forall x \in \partial(T_1^{-n}R), \ n > 0 \tag{1.11}$$

for some θ with $(1+\gamma)(1-\theta) > 1$.

Remark 1.13. *The conditions in Theorem B.i) mean that DT_p has at least one eigenvalue with absolute value greater than 1.*

Remark 1.14. *In Theorem C. ii), the cones are not in general continuous with x at the point p, though they are continuous in (t, ϕ). (See Example 1.)*

Remark 1.15. *The condition in Theorem C.ii) part (c) implies that under DT, vectors in the cone $\mathcal{C}_x$ expands faster than that in $\mathcal{C}'_x$. To see this, let us first recall that if a 2×2 matrix has two eigenvectors v_1 and v_2 with corresponding eigenvalues $\lambda_1 > \lambda_2 > 0$ respectively, then under iterations all vectors, except for v_2, move toward v_1, up to a coefficient. So if two vectors are close to v_1, then the angle between these two vectors decreases under iterations of A, and if we replace v_1 by v_2, then the angle between them increases. In our case, cone invariance in part (a) corresponds the fact that DT_x has two eigenvectors and (1.10) implies that the angle between v and v' becomes smaller under DT_x. (See Lemma 4.2.)*

Remark 1.16. *If we write $DT(x) = T_0(x) + T_\gamma(x) + T_h(x)$, where $T_0 = DT_p$, T_γ satisfies $T_\gamma(tx) = t^\gamma T_\gamma(x)$ $\forall t > 0$ and $|T_h(x)| = O(|x|^{\gamma'})$, $\gamma' > \gamma$, then the construction of the cones $\{\mathcal{C}_x\}$ and $\{\mathcal{C}'_x\}$ depends substantially on T_γ as x near p: this is explicitly seen in the proof of Lemma 4.3. So it is easy to get uniformity near $t = 0$.*

2 Examples

Now we give three examples of maps for which all the assumptions of Theorem A can be checked. Since we allow discontinuities, it is easy to construct a map that satisfies Assumption 1 to 3 and 4(a). So our examples concentrate mostly on the local behavior of the map in the neighborhood of the indifferent fixed point $p = 0$, although a complete description is provided for example 1.

In the latter we show that the map has unbounded distortion, and the conditions in part ii) of Theorem B and C can be verified, and then that the map can have both finite and infinite components for the absolutely continuous invariant measure.

Example 1. *We let $M \subset \mathbb{R}^2$ and near the fixed point $p = (0,0)$, the map T has the form*

$$T(x,y) = \big(x(1+x^2+y^2),\ y(1+x^2+y^2)^2\big) \tag{2.1}$$

up to order $O(|z|^4)$, where $z = (x,y)$ and $|z| = \sqrt{x^2+y^2}$.

It is easy to see that

$$DT(x,y) = \begin{pmatrix} 1+3x^2+y^2+O(|z|^4) & 2xy+O(|z|^4) \\ 4xy+O(|z|^4) & 1+2x^2+6y^2+O(|z|^4) \end{pmatrix}, \tag{2.2}$$

and

$$\det DT(x,y) = 1+5x^2+7y^2+O(|z|^4), \tag{2.3}$$

Note that in this example, T is locally injective and T^{-1} will denote its inverse.

Unbounded distortion
We begin to show that the distortion is unbounded even away from $p = 0$ in the sense that there are uncountably many points z such that for any neighborhood V of z, we can find $\hat{z} \in V$ such that the ratio

$$|\det DT_1^{-n}(z)|/|\det DT_1^{-n}(\hat{z})| \tag{2.4}$$

is unbounded as $n \to \infty$.
Take $z' = (x_0, 0)$ and denote $z'_n = T^{-n}z'$. By Lemma 3.1 in the next section, we have $|z'_n| \sim \dfrac{1}{\sqrt{2n}}$, where $a_n \sim b_n$ means $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$. Hence by (2.3) and Lemma 3.2 with $t_i = z_i$, $r(t_i) = |detDT(z_i)|$, $C' = 5$ and $\gamma C = 2$, we get $|\det DT^{-n}(z')| \le \dfrac{D'}{n^{5/2}}$ for some $D' > 0$. On the other hand if we take $z'' = (0, y_0)$ and denote $z''_n = T^{-n}z''$, then $|z''_n| \sim \dfrac{1}{\sqrt{4n}}$ and $|\det DT^{-n}(z'')| \ge \dfrac{D''}{n^{7/4}}$ for some $D'' > 0$. So $\dfrac{|\det DT^{-n}(z'')|}{|\det DT^{-n}(z')|} \to \infty$ as $n \to \infty$.

We take a curve from z' to z'' that does not contain the origin. If for every z on the curve, there is a neighborhood V such that for all $\hat{z} \in V$, the ratio in (2.4) is bounded for all $n > 0$, then the ratio $|\det DT^{-n}(z'')|/|\det DT^{-n}(z')|$ should be bounded. This contradicts the above fact. So we know that there are some points on the curve at which distortion is unbounded. By moving z' and z'', we can get uncountably many pairwise disjoint curves and therefore we get what we need.

Validity of Assumption 4(b)
For any $z = (x, y)$, we put again $z_n = T^{-n}z$.

Note that

$$|z|(1 + |z|^2 + O(|z|^4)) \le |Tz| \le |z|(1 + 2|z|^2 + O(|z|^4)),$$

or

$$|z_n|\big(1 + |z_n|^2 + O(|z_n|^4)\big) \le |z_{n-1}| \le |z_n|\big(1 + 2|z_n|^2 + O(|z_n|^4)\big).$$

So by Lemma 3.1, we have

$$\frac{1}{\sqrt{4(n+k)}} + O(n^{-\beta'}) \le |z_n| \le \frac{1}{\sqrt{2(n+k)}} + O(n^{-\beta'}), \tag{2.5}$$

for some integer k, where $\beta' > 1/2$.

Since (2.3) implies that $|\det DT(z)|^{-1} \le 1 - 5|z|^2 + O(|z|^4)$, by (2.5) and Lemma 3.2 we get

$$|\det DT^{-n}(z)| \le Dn^{-5/2}. \tag{2.6}$$

Also by (2.2),

$$DT^{-1}(x, y) = \begin{pmatrix} 1 - 3x^2 - y^2 + O(r^4) & -2xy + O(r^4) \\ -4xy + O(r^4) & 1 - 2x^2 - 6y^2 + O(r^4) \end{pmatrix}.$$

So $\|DT^{-1}(z)\| \le 1 - |z|^2 + O(|z|^4)$, hence by Lemma 3.2,

$$\|DT^{-n}(z)\| \le D'n^{-1/2} \tag{2.7}$$

for some $D' > 0$. Now by (2.6), (2.7) and (1.9), we know that $\delta = 5/2$ and $\tau = 1/2$. Since $m = 2$ and $\gamma = 2$, we have (1.8) if $\alpha = 1/2$. By Theorem B.ii), T satisfies Assumption 4(b).

Validity of Assumption 4(c)
It is easy to check that we can apply Theorem C.i) as we will do in Example 2 below. However, we use this map to show how to apply Theorem C.ii).

Notice that if we take two vectors $v_0 = (x, y)^*$ and $v_0' = (y, -x)^*$ at the tangent plane of $z = (x, y)$, where the asterisk denotes transpose, then by (2.2) we have

$$DT_z(v_0) = \begin{pmatrix} x + 3x^3 + 3xy^2 + O(|z|^5) \\ y + 6x^2y + 6y^3 + O(|z|^5) \end{pmatrix},$$

$$DT_z(v_0') = \begin{pmatrix} y + x^2y + y^3 + O(|z|^5) \\ -x - 2x^3 - 2xy^2 + O(|z|^5) \end{pmatrix}.$$

This means that $|DT_z(v_0')| < |DT_z(v_0)|$. We define $\mathcal{C}_z$ at each point z as the cone bounded by lines generated by vectors $3v_0 + 2v_0'$ and $3v_0 - 2v_0'$ and containing v_0, and define $\mathcal{C}_z'$ as the cone bounded by lines generated by vectors $3v_0' + 2v_0$ and $3v_0' - 2v_0$ and disjoint with $\mathcal{C}_z$. We can check that part (a) and (b) in Theorem B.ii) are satisfied. Also we can check that for all unit vector $v' \in \mathcal{C}_z'$, $|DT_z(v')| \le |Tz|^{2.5}/|z|^{2.5}$. So if we take R in such a way that the tangent lines of $\partial(T_1^{-n_0}R)$ are in the cones $\mathcal{C}'$ for some $n_0 \ge 0$, then we use the fact $DT^{-1}(\mathcal{C}') \subset \mathcal{C}'$ to get that part (c) is satisfied for all $n \ge n_0$ with $1 - \theta = 2/5$.

Coexistence of finite and σ-finite components
We now arrange this map in such a way that the absolute continuous invariant measure μ has a finite and a σ-finite ergodic components simultaneously, and both contain the same indifferent fixed point $p = 0$ in their supports. We assume that our map satisfies Assumptions 1 to 3 and 4(a), besides the condition 4(b) and 4(c) checked above. We also assume that there is a partition of $M = \{M_1, M_2\}$ such that for $i = 1, 2$, $TM_i = M_i$ and for any ball $B_\varepsilon(x)$ in M_i, there exists an integer N such that $T^N B_\varepsilon(x) = M_i$, and

$$\{z = (x, y) \in B_1(p) : y < x^2\} \subset M_1, \quad \{z = (x, y) \in B_1(p) : y > x^2\} \subset M_2.$$

This is possible since it is easy to check that $T\Gamma \cap B_1(p) = \Gamma$, where $\Gamma = \{(x, y) \in B_1(p) : y = x^2\}$. The assumptions just required insure the applicability of Theorem A. Since both M_1 and M_2 are invariant sets, T has absolutely continuous invariant measures μ_1 and μ_2 with respect to the Lebesgue measure restricted to M_1 and M_2 respectively. Now we show $\mu_1 M_1 < \infty$ and $\mu_2 M_2 = \infty$.

For this purpose we may assume that $R = B_1(p)$. By (2.5), we know that $T_1^{-n}R \subset B_{2/\sqrt{2n}}(p)$ for all large n. So

$$\nu(T_1^{-n}R \cap M_1) \le \nu\{(x, y) : x^2 + y^2 \le \frac{4}{2n},\ |y| \le |x|^2\} \le C\Big(\frac{4}{2n}\Big)^{3/2}$$

for some $C > 0$. It follows that $\sum_{n=1}^{\infty} \nu(T_1^{-n}R \cap M_1) < \infty$. Applying Theorem A to the system $T : M_1 \to M_1$, we get that $\mu_1 M_1 \le \infty$.

Also, by (2.5), we have that $T_1^{-n}R \supset B_{1/2\sqrt{4n}}(p)$ for all large n. Hence it is easy to see that $\nu(T_1^{-n}R) \ge \pi/16n$ and therefore $\sum_{n=1}^{\infty} \nu(T_1^{-n}R) = \infty$. Since $\nu\big(T_1^{-n}R \cap M_1\big) + \nu\big(T_1^{-n}R \cap M_2\big) = \nu\big(T_1^{-n}R\big)$, we get $\sum_{n=1}^{\infty} \nu(T_1^{-n}R \cap M_2) = \infty$. So we have $\mu_2 M_2 = \infty$.

Full construction of the map
We now show how to arrange this map in order to verify all the assumptions required by Theorem A.

Let $M = \{(x, y) \in \mathbb{R}^2 : |x|, |y| \le 1000\}$. Take a partition $\{U_1, \cdots, U_K\}$ of M such that

(i) $U_1 = \{(x, y) \in \mathbb{R}^2 : |x|, |y| \le 10\}$;

(ii) $\operatorname{diam} U_i \le 10$ if $i \ne 1$;

(iii) ∂U_i are piecewise smooth curves; and

(iv) every point $x \in M$ is contained in at most 3 ∂U_is.

Let $R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le (3+\epsilon_1)^2\}$, where ϵ_1 was used to define $s(x)$ in Section 1.

We define $T : M \to M$ such that

(a) on the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 4^2\}$, T has the form

$$T(x, y) = \big(x(1 + x^2 + y^2),\ y(1 + x^2 + y^2)^{\tau(x,y)}\big)$$

where τ is a smooth function decreasing with $x^2 + y^2$ such that $\tau(x, y) = 2$ if $x^2 + y^2 \le 1$; $\tau(x, y) = 1$ if $2 \le x^2 + y^2 \le 3.5$. So in the neighborhood of $(0, 0)$, T has the form (2.1);

(b) $T_i = T|_{U_i}$ is a C^2 map for all i, and can be extended to an ϵ neighborhood of U_i;

(c) $s(x) < 0.1, \forall x \notin R$;

(d) $R \notin T(\partial U_i)$ for any i.

Note that on the boundary of R, $s(x)^{-1} \ge 10$. In fact, observe that near the boundary of R,

$$T(x, y) = \big(x(1 + x^2 + y^2),\ y(1 + x^2 + y^2)\big).$$

It is easy to see by the choice of ϵ_1 that at the point $x = (3+\epsilon_1, 0)$, $s(x)^{-1} \ge 10$. Observe that DT is the same at every point of ∂R, up to a rotation. So we have that $s(x)^{-1} \ge 10, \forall x \in \partial R$. This shows that such systems exist.

It is obvious that Assumption 1 to 2 are satisfied.

Note that $TR = \{(x, y) : x^2 + y^2 \le (3 + \varepsilon_1)(1 + (3 + \varepsilon_1)^2)\} \subset U_1$. So by (d) the requirements for R in Assumption 3 are satisfied.

Assumption 3(a) is obvious.

Since for $x \in TR \setminus R$, $|T_1^{-n} x|$ has the order $n^{-1/2}$, we get $\|DT_1^{-1}(T_1^{-n} x)\| \le 1 - c/n$ for some $c > 0$. So we have $\|DT_1^{-n}(x)\| \to 0$. By Remark 1.5, Assumption 3(c) follows.

Assumption 4(a) follows from (b).

Now we verify Assumption 3(b). By (c) we have $s \le 0.1$. Note that $\gamma_2 = \pi$, $\gamma_1 = 2$. By (iv), we can choose $Y = 3$. Then

$$\frac{4s}{1-s} Y \frac{\gamma_1}{\gamma_2} \le \frac{0.4}{0.9} 3 \frac{2}{\pi} = \frac{8}{3\pi} \le 0.85.$$

So by Remark 1.3, we make Assumption 3(b) holds.

The next example shows how to use part i) in Theorem B and C to verify Assumption 4(b) and 4(c).

Example 2. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by*

$$T(x, y, z) = \big(x(1 + x^2 + y^2 + z^2), y(1 + x^2 + y^2 + z^2)^2, z(2 + x^2 + y^2 + z^2)^3\big)$$

as (x, y, z) near the origin.

Note that by an argument similar to that given in the preceding example, we can prove that for this map the distortion is unbounded away from the origin.

Since $\det DT_{(0,0,0)} = 2$ and T is C^∞ near the origin, by Theorem B.i), Assumption 4(b) is satisfied.

Let $\mathcal{D}_i$, $i = 1, \cdots, 8$, be the eight octants in $\mathbb{R}^3$. Clearly all the requirements in Theorem C.i) are satisfied. So we get Assumption 4(c) as well.

The last example shows that our results cover an open set in the space of piecewise expanding maps T with an indifferent fixed point p such that $DT(p)$ is an isometry, and $s(x) > 1$ $\forall x \neq p$.

Example 3. *Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be given by*

$$T(x) = Ex + H(x, x, x) + O(|x|^4), \tag{2.8}$$

where $x \in \mathbb{R}^m$ is a column vector, E is an $m \times m$ orthogonal matrix, $H : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a linear map close to the map $H_0(x, x, x) = x^T x(Ex)$, and x^T denote the transpose of x.

We first consider the case $H = H_0$. Then

$$T(x) = Ex(1 + x^T x) + O(|x|^4), \tag{2.9}$$

and

$$DT(x) = E(I + |x|^2 I + 2xx^T) + O(|x|^3),$$

where I denote the $m \times m$ identity matrix. It is easy to see that

$$\begin{aligned} &|\det DT(x)| = 1 + (m + 2)|x|^2 + O(|x|^3), \\ &\|DT^{-1}(x)\|^{-1} = 1 + |x|^2 + O(|x|^3). \end{aligned}$$

So if H is close to H_0, then

$$\begin{aligned} |\det DT_p(x)|^{-1} \quad &\leq 1 - \bar{C}_1'|x|^2 + O(|x|^3), \\ \|DT_p^{-1}p(x)\| \quad &= 1 - \hat{C}_1'^{|} x|^2 + O(|x|^3) \end{aligned}$$

for some $\bar{C}_1'$ close to $m + 2$ and $\hat{C}_1'$ close to 1.

Also note that if H is close to H_0, then

$$|T^{-1}(x)| \geq |x|(1 - C'|x|^2 + O(|x|^3)),$$

for some C' close to 1. So if we take x in a neighborhood of p and denote $x_n = T^{-n}x$, then by Lemma 3.1,

$$|x_{n-1}| \geq \frac{1}{(2C(n+k))^{1/2}} + O\big(\frac{1}{(n+k)^{\beta'}}\big)$$

for some $k > 0$ and $\beta > 1/2$. By Lemma 3.2, we get that

$$|\det DT_1^{-n}(x)| \leq \bar{D}\Big(\frac{k}{n+k}\Big)^{\bar{C}_1'/2C}, \quad \|DT_1^{-n}(x)\| \leq \hat{D}\Big(\frac{k}{n+k}\Big)^{\hat{C}_1'/2C} \tag{2.10}$$

for some $\bar{D}, \hat{D} > 0$. Hence, we get (1.9) for $\delta = \bar{C}_1'/2C$ and $\tau = \hat{C}_1'/2C$ which are close to $1+m/2$ and $1/2$ respectively. Since $\gamma = 2$, we get (1.8) if α is close to 0. The requirements of Thereom B.ii) are satisfied. This verifies Assumption 4(b).

For each x we let E_x denote the one dimensional subspace spanned by the vector x, and E_x' be the orthogonal complement of E. Then we define the cones

$$\mathcal{C}_x = \{u + u' : u \in E_x, u' \in E_x', |u'| \leq |u|/2\}$$
$$\mathcal{C}_x' = \{u + u' : u \in E_x, u \in E_x', |u| \leq |u'|/2\}$$

Note that if $H = H_0$, then we have $|DT(x)(u)| = 1 + 3|x|^2|u|$ for $u \in E_x$ and $|DT(x)(u')| = 1 + |x|^2|u'|$ for $u' \in E_x'$. Using this fact we can check that all requirements in Theorem C.ii) are satisfied. Also note that DT change continuously with the third order term H of T. So if H is close to H_0, then the requirements of Theorem C.ii) are also satisfied. This verifies Assumption 4(c) for these T.

Remark 2.1. *Consider the set $\mathcal{S}$ of all C^4 maps T defined in a neighborhood of 0 in $\mathbb{R}^m$ satisfying $T(0) = 0$, $DT(0) = E$, an isometry map, and $s(x) < 1$ if $x \neq 0$, where $s(x)$ is defined before Assumption 3. The condition $s(x) < 1$ implies $D^2T(p) = 0$. Hence the Taylor expression of T has the form (2.8), where $H = 6D^3T$.*

If $T_a, T_b \in \mathcal{S}$ are close in C^3 topology, then the corresponding H_a and H_b are close. So the example implies that our results cover an open subset of $\mathcal{S}$ that containing the map given in (2.9).

3 Proof of Theorem B

We first prove some lemmas.

For $\gamma > 0$, let $\beta = 1/\gamma$.

Lemma 3.1. *If*

$$t_{n-1} \geq t_n + Ct_n^{1+\gamma} + O(t_n^{1+\gamma'}) \qquad \forall n > 0, \tag{3.1}$$

where $\gamma' > \gamma$, then for all large n,

$$t_n \leq \frac{1}{(\gamma C(n+k))^{\beta}} + O\big(\frac{1}{(n+k)^{\beta'}}\big) \qquad \forall n > 0 \tag{3.2}$$

for some $\beta' > \beta$ and $k \in \mathbb{Z}$. The result remains true if we exchange "$\leq$" and "$\geq$". Therefore, if (3.1) becomes an equality, then so does (3.2).

Proof: We claim that if

$$t_{n-1} \geq t_n + C t_n^{1+\gamma} + C' t_n^{1+\gamma'}, \tag{3.3}$$

for some large n and

$$t_n^\gamma \geq \frac{1}{\gamma C n}\Big(1 + \frac{1}{n^{\delta'}}\Big) \tag{3.4}$$

for some $\delta' > 0$, then

$$t_{n-1}^\gamma \geq \frac{1}{\gamma C(n-1)}\Big(1 + \frac{1}{(n-1)^{\delta'}}\Big).$$

This gives the results since we can choose an integer k such that for some large $n_0 > 0$,

$$t_n^\gamma \leq \frac{1}{\gamma C(n_0+k)}\Big(1 + \frac{1}{(n_0+k)^{\delta'}}\Big).$$

By relabelling the indices, the claim implies (3.2) for all $n \geq n_0$.

Now we prove the claim. Denote $\gamma_n = \gamma\big(1 + n^{-\delta'}\big)^{-1}$. By (3.3) and (3.4),

$$t_{n-1}^\gamma \geq {t_n}^\gamma\big(1 + C{t_n}^\gamma + C'{t_n}^{\gamma'}\big)^\gamma \geq \frac{1}{Cn\gamma_n}\Big(1 + \frac{C}{Cn\gamma_n} + \frac{C'}{(Cn\gamma_n)^{\gamma'/\gamma}}\Big)^\gamma.$$

To prove the lemma we only need to show that

$$\frac{1}{n\gamma_n}\Big(1 + \frac{1}{n\gamma_n} + \frac{C'}{(Cn\gamma_n)^{\gamma'/\gamma}}\Big)^\gamma \geq \frac{1}{(n-1)\gamma_{n-1}},$$

or, equivalently,

$$\frac{n-1}{n}\Big(1 + \frac{1}{n\gamma} + \frac{1}{n^{1+\delta'}\gamma} + \frac{C'}{(Cn\gamma_n)^{\gamma'/\gamma}}\Big)^\gamma \geq \frac{\gamma_n}{\gamma_{n-1}} = \frac{1 + (n-1)^{-\delta'}}{1 + n^{-\delta'}}.$$

Take $\delta' < \min\{1, \gamma'/\gamma - 1\}$. Then $(n\gamma_n)^{-(\gamma'/\gamma)}$ is of higher order. We can check that as $n \to \infty$, the left side of the inequality is like $1 + n^{-(1+\delta')}$ and the right side is like $1 + \delta' n^{-(1+\delta')}$. Since $\delta' < 1$, the right side is smaller as n is large. □

Lemma 3.2. *If for all $n > 0$, t_n satisfies (3.2), and $r(t_n) \leq 1 - C' t_n^\gamma + O(t_n^{1+\gamma'})$, where $C' > 0$, then there exists $D > 0$ such that for all $k_0 \geq k$,*

$$\prod_{i=k_0-k}^{n+k_0-k} r(t_i) \leq D\Big(\frac{k}{n+k}\Big)^{C'/\gamma C}. \tag{3.5}$$

The result remains true if we replace "$\leq$" by "$\geq$" in all three inequalities.

Proof: Note that

$$r(t_n) \le 1 - \frac{C'}{\gamma C n} + O\Big(\frac{1}{n^{1+\gamma'}}\Big) = \Big(1 - \frac{1}{n}\Big)^{\frac{C'}{\gamma C}} \cdot \Big(1 + O\big(\frac{1}{n^{1+\gamma'}}\big)\Big),$$

where $\gamma' > 0$. Then we take product. $\square$

Lemma 3.3. *Let $\theta \in (0,1)$ and $\bar{C}_1', \bar{C}_2, \bar{D}_1 > 0$, and let $\tilde{R} \subset \mathbb{R}^m$ be a bounded region containing the origin. Suppose the map $T : \tilde{R} \to \mathbb{R}^m$ is injective with $T^{-1}\tilde{R} \subset \tilde{R}$ and satisfies*

$$d(Tx, Ty) \ge (1 + \bar{C}_1'|x|^\gamma) d(x,y), \tag{3.6}$$

$$\log\Big|\frac{\det DT(x)}{\det DT(y)}\Big| \le \bar{C}_2 |x|^{\gamma-1} d(x,y) \tag{3.7}$$

for all $x, y \in \tilde{R}$ with $d(x,y) \le |x|/2$. Then there exists $J' > 0$ such that for all $x, y \in T\tilde{R}$ with

$$d(x_i, y_i)^{1-\theta} \le \bar{D}_1 |x_i|, \quad i = 1, \cdots, n, \tag{3.8}$$

where $x_i = T^{-i}x$ and $y_i = T^{-i}y$, we have

$$\log\Big|\frac{\det DT^n(x_n)}{\det DT^n(y_n)}\Big| \le J' d(x,y)^\theta. \tag{3.9}$$

Proof: We prove by induction that for all $i = 1, \cdots, n$,

$$\log\Big|\frac{\det DT^i(x_n)}{\det DT^i(y_n)}\Big| \le J' d(x_{n-i}, y_{n-i})^\theta. \tag{3.10}$$

For $i = 1$, by (3.7), (3.8) and (3.6), we have

$$\log\Big|\frac{\det DT(x_n)}{\det DT(y_n)}\Big| \le \bar{C}_2 \bar{D}_1 |x_n|^\gamma d(x_n, y_n)^\theta \le \bar{C}_2 \bar{D}_1 |x_{n-1}|^\gamma d(x_{n-1}, y_{n-1})^\theta.$$

So if $J' \ge \sup\{\bar{C}_2 \bar{D}_1 |x|^\gamma : \ x \in \tilde{R}\}$ then the right side of the inequality is less than $J' d(x_n, y_n)^\theta$ because $|x_n| \le |x|$.

Suppose (3.10) is true up to $i = k-1$. Then similarly we have

$$\begin{aligned}
&\log\Big|\frac{\det DT^k(x_n)}{\det DT^k(y_n)}\Big| \le \log\Big|\frac{\det DT^{k-1}(x_n)}{\det DT^{k-1}(y_n)}\Big| + \log\Big|\frac{\det DT(x_{n-k+1})}{\det DT(y_{n-k+1})}\Big| \\
\le\ & J' d(x_{n-k+1}, y_{n-k+1})^\theta + \bar{C}_2 |x_{n-k+1}|^{\gamma-1} d(x_{n-k+1}, y_{n-k+1}) \\
=\ & J'\Big(1 + \frac{\bar{C}_2 \bar{D}_1}{J'} |x_{n-k+1}|^\gamma\Big) \cdot \frac{d(x_{n-k+1}, y_{n-k+1})^\theta}{d(x_{n-k}, y_{n-k})^\theta} \cdot d(x_{n-k}, y_{n-k})^\theta \\
\le\ & J'\Big(1 + \frac{\bar{C}_2 \bar{D}_1}{J'} |x_{n-k+1}|^\gamma\Big) \cdot \frac{1}{(1 + \bar{C}_1'|x_{n-k+1}|^\gamma)^\theta} d(x_{n-k}, y_{n-k})^\theta.
\end{aligned}$$

Clearly if J' is large enough, then the right side is bounded by $J'd(x_{n-k}, y_{n-k})^\theta$. We get (3.10) for $i = k$. $\square$

We are finally ready to prove the theorem of this section.

Proof of Theorem B:
Part i). We assume that T is $C^{1+\hat{\alpha}}$ and $|\det DT| \geq \kappa^{-1} > 1$ on TR. We also regard $\hat{\alpha} \leq 1$. So there exist $c_1 > 0$ such that

$$\frac{|\det DT_1^{-1}(y)|}{|\det DT_1^{-1}(x)|} \leq 1 + c_1 d(x,y)^{\hat{\alpha}}$$

for all $x, y \in TR$. Let $x_i = T_1^{-i}x$ and $y_i = T_1^{-i}y$. Clearly, $d(x_i, y_i) \leq d(x,y)$. So if $d(x,y) \leq \varepsilon$ and $0 < n \leq N$, then

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|} \leq \left(1 + c_1 d(x,y)^{\hat{\alpha}}\right)^n \leq \left(1 + c_1 \varepsilon^{\hat{\alpha}}\right)^N. \tag{3.11}$$

Also, there exists $C > 0$ such that for any $y \in B_\varepsilon(R_0)$, $|\det DT_1^{-n}(y)| \leq C\kappa^n$. Hence,

$$\sum_{n=N}^{\infty} \sup_{y \in B_\varepsilon(x)} |\det DT_1^{-n}(y)| \leq \frac{C\kappa^N}{1-\kappa}.$$

Let $b > 0$ be given.
Consider the function

$$\sigma(\varepsilon) = \frac{(1 + c_1\varepsilon^{\hat{\alpha}})^{N_0 - c_2 \log \varepsilon}}{1 + J\varepsilon^\alpha},$$

where $N_0 = 1 + \log(C^{-1}b(1-\kappa))/\log \kappa$ and $c_2 = -(m+\alpha)/\log \kappa$. Since $\lim_{\varepsilon \to 0} (1 + c_1\varepsilon^{\hat{\alpha}})^{N_0 - c_2 \log \varepsilon} = 1$, we have $\lim_{\varepsilon \to 0} \sigma(\varepsilon) = 1$. Note that if

$$(N_0 - c_2 \log \varepsilon)\hat{\alpha} c_1 \varepsilon^{\hat{\alpha}-1} \cdot (1 + J\varepsilon^\alpha) - \alpha J \varepsilon^{\alpha-1} \cdot (1 + c_1\varepsilon^{\hat{\alpha}}) < 0, \tag{3.12}$$

then $\sigma'(\varepsilon) < 0$. Since $\hat{\alpha} > \alpha$, the first term in (3.12) is of higher order. So we can choose $J > 0$ and $\varepsilon_4 > 0$ such that for all $\varepsilon \in (0, \varepsilon_4]$, (3.12) holds and therefore $\sigma(\varepsilon) \leq 1$.

Now for each $\varepsilon \in (0, \varepsilon_4]$, we take $N = N(\varepsilon)$ as the integer part of $N_0 - c_2 \log \varepsilon$. Clearly, for such N we have

$$\frac{C\kappa^N}{1-\kappa} \leq b\varepsilon^{m+\alpha}.$$

So the second inequality in Assumption 4(b) is true. For the first inequality, note that

$$(1 + c_1\varepsilon^{\hat{\alpha}})^N \leq (1 + c_1\varepsilon^{\hat{\alpha}})^{N_0 - c_2 \log \varepsilon} \leq 1 + J\varepsilon^\alpha.$$

Then by (3.11) we get what we need.

Part ii). Let us put $\beta = 1/\gamma$ and $\theta = \alpha$. Take $\rho > 0$ such that

$$\frac{\beta}{1-\theta} - \tau < \rho < \frac{\delta - 1}{m+\alpha}. \tag{3.13}$$

Let $b > 0$ be given.

Note that by Lemma 3.1, (1.5) implies that there exists $\bar{C}_0 > 0$ such that for any $x \in R_0$, $|x_n| \geq \frac{1}{(\bar{C}_0 n)^\beta}$. Take $N_b \geq L$ such that for all $n \geq N_b$,

$$b^{-\frac{1}{m+\alpha}} \Big(\sum_{k=n}^{\infty} \frac{C_\delta}{k^\delta} \Big)^{\frac{1}{m+\alpha}} < \frac{1}{n^\rho} < \frac{1}{(n-1)^\rho} < \frac{1}{2C_\tau \bar{C}_0^{\frac{\beta}{1-\theta}} n^{\frac{\beta}{1-\theta} - \tau}}, \tag{3.14}$$

where C_δ and C_τ are as in (1.9). The inequality is possible because of (3.13).

Note that (1.6) and (1.7) imply (3.6) and (3.7) respectively. By Lemma 3.3 we can take $J' > 0$ such that (3.9) holds for any $x \in R_0$, $n > 0$ whenever (3.8) holds with $\bar{D}_1 = 1$ for all x_i, y_i, $i = 1, \cdots, n$.

Take $\varepsilon_4' > 0$ such that for all x, y with $x \in R_0$, $d(x,y) \leq \varepsilon_4'$, $n = 1, \cdots, N_b$, we have $d(x_n, y_n)^{1-\theta} \leq |x_n|$. By the choice of J', (3.9) holds for all $1 \leq n \leq N_b$.

Then we take $\varepsilon_4 = \min\{\varepsilon_4', 1/N_b^\rho\}$, and $J > 0$ such that $e^{J' \varepsilon_4^\theta} \leq 1 + J\varepsilon_4^\theta$.

We show that J and ε_4 satisfies the requirement. Let $\varepsilon \in (0, \varepsilon_4]$. Take $N = N(\varepsilon) > N_b$ such that

$$\frac{1}{N^\rho} \leq \varepsilon < \frac{1}{(N-1)^\rho}.$$

By the first inequality of (1.9) and (3.14),

$$\sum_{k=N}^{\infty} \sup_{y \in B_\varepsilon(x)} |\det DT^{-k}(y)| \leq \sum_{k=N}^{\infty} \frac{C_\delta}{k^\delta} \leq b \cdot \frac{1}{N^{\rho(m+\alpha)}} \leq b\varepsilon^{m+\alpha}.$$

On the other hand, if $x \in R_0$ and $d(x,y) \leq \varepsilon$, then by the last inequality of (1.9) and (3.14), for any $N_b < n \leq N$,

$$d(x_n, y_n) \leq \frac{2C_\tau}{n^\tau} \varepsilon \leq \frac{2C_\tau}{n^\tau} \frac{1}{(N-1)^\rho} \leq \frac{1}{\bar{C}_0^{\frac{\beta}{1-\theta}} n^{\frac{\beta}{1-\theta}}} \leq |x_n|^{\frac{1}{1-\theta}}.$$

So we know that (3.9) holds for all $0 \leq n \leq N$. Then by the choice of J and the fact $\theta = \alpha$,

$$\Big| \frac{\det DT^n(x_n)}{\det DT^n(y_n)} \Big| \leq e^{J' d(x,y)^\theta} \leq e^{J' \varepsilon^\alpha} \leq 1 + J\varepsilon^\alpha.$$

This is what we need. $\square$

4 Proof of Theorem C

The proof consists again of two parts.
Proof of Theorem C:
Part i). For $x \in \partial R$, denote

$$\mathcal{D}(x) = \{z \in R_0 : \ x \prec z \prec Tx\}.$$

Clearly the collection $\{\mathcal{D}(x) : \ x \in \partial R \cap \mathcal{D}_i\}$ form a cover of $\mathcal{D}_i \cap R_0$. Note that by the meaning of $x \prec Tx$, every $\mathcal{D}(x)$ contains an open set in $\mathcal{D}_i \cap R_0$. So we can construct a finite or countable partition ξ of R_0 such that every element of ξ belongs to the closure $\overline{\mathcal{D}(x)}$ of some $\mathcal{D}(x)$.

Note that for any x,

$$\frac{|\det DT_1^{-n}(x)|}{|\det DT_1^{-n}(Tx)|} = \frac{|\det DT(x)|}{|\det DT(x_n)|} \leq |\det DT(x)|$$

is always bounded. So for any $y, z \in \overline{\mathcal{D}(x)}$, we have

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(z)|} \leq \frac{|\det DT_1^{-n}(x)|}{|\det DT_1^{-n}(Tx)|} \leq |\det DT(x)|.$$

Hence (1.4) follows. Obviously we can arrange the partition ξ in such a way that (1.3) also holds. Therefore ξ is a desired partition for any n.
Part ii). First, we take $\theta > 0$ such that

$$DT_x(v') \leq (|Tx|/|x|)^{1/(1-\theta)}$$

for all $x \in \partial(T_1^{-n}R)$ and $v' \in E_x(\partial(T_1^{-n}R))$. This is possible because of the assumption stated in part (d) of Theorem C.ii). So for any $n > 0$, if we take $x, y \in \partial R_0$ such that $d(x_n, y_n) \leq \bar{D}_1 |x_n|^{1/(1-\theta)}$, we have

$$d(x_i, y_i) \leq \bar{D}_1 |x_i|^{1/(1-\theta)} \qquad \forall i = 1, \cdots, n. \tag{4.1}$$

By Lemma 3.3, we get that there exists $I_1 > 0$ such that

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|} \leq I_1. \tag{4.2}$$

That is, (1.4) holds for all such x, y.

We construct $\xi = \xi_n$. Note that we only need do it for n sufficiently large. Since the family of cones $\mathcal{C}'_x$ are uniformly continuous in $x = (t, \phi)$, we can find $t_0 > 0$ such that for any $x, y \in TR$ with $d(x, y) \leq t_0$, the Hausdorff distance between $\mathcal{C}'_x$ and $\mathcal{C}'_y$ is less than $\theta_0/2$. Then we take $N > 0$ large enough such that for any $x \in R_0$ and $n > N$, $|x_n| \leq t_0$. Note that for any x, the position vector from p to x, denoted by u_x, is contained in $\mathcal{C}_x$. By part (a) and (d) in the conditions of the theorem we know that at $x \in T_1^{-n}(\partial R_0)$, $\mathcal{C}'_x$ contains the tangent plane of the surface. Hence, if $v' \in E_x(T_1^{-n}(\partial R_0)$, then the angle between u_x and v', denoted by $\angle(u_x, v')$, is larger than θ_0, and therefore for any $v' \in E_y(T_1^{-n}(\partial R_0)$, we have $\angle(u_x, v') \geq \theta_0/2$, whenever $y \in T_1^{-n}(\partial R_0)$ with $d(x,y) \leq t_0$. So for any $x, y \in T_1^{-n}(\partial R_0)$ with $d(x,y) \leq t_0$, we have $d_S(x,y) \leq d(x,y)/\sin(\theta_0/2)$, where $d_S(\cdot,\cdot)$ is the distance restricted to the surfaces $\{T_1^{-n}(\partial R_0)\}$. This means that we can take a partition $\xi^{(n)}$ on $T_1^{-n}(\partial R_0)$ such that every element of $\xi^{(n)}$ is contained in a ball of radius $|x_n|^{1/(1-\theta)}$ and containing a ball of radius $|x_n|^{1/(1-\theta)}/10m\sin(\theta_0/2)$, with respect to the metric on $T_1^{-n}(\partial R_0)$, and these elements are close to $(m-1)$ dimensional disks. Denote $\xi' = T^n\xi^{(n)}$. Clearly, it is a partition of ∂R_0. Then we can take a partition ξ of R_0 whose elements has the form $\cup_{x\in A'}\mathcal{F}_x \cap R_0$, where A' is an element of ξ', and $\mathcal{F}_x$ is given in Lemma 4.1.

Now we prove that ξ satisfies (1.3) and (1.4). Condition (1.6) implies $\|DT(p)\| = 1$. We first consider the case that $DT(p) = \mathrm{id}$.

By (1.5), we know that $d(x, Tx) \leq C|x|^{1+\gamma}$ for some $C > 0$. So the "width" of the annulus $T_1^{-i}(B_{\varepsilon_5}(R_0))$ is bounded by $C'|T_1^{-i}x|^{1+\gamma}$ for some $C' > 0$. By part (b) and (d) of the condition in the theorem, for $0 < \varepsilon \leq \varepsilon_5$, $x \in T_1^{-i}(\partial B_\varepsilon(R_0))$, the angle between the tangent space of $T_1^{-i}(\partial B_\varepsilon(R))$ and the position vector u_x is larger than θ_0. So the length of the curve $\mathcal{F}_{T_1^{-i}x} \cap T_1^{-i}B_{\varepsilon_5}(R_0)$ is bounded by $C|T_1^{-i}x|^{1+\gamma}$ for some $C \geq C'$. Hence, for any $x, y \in B_{\varepsilon_5}(R_0)$ with $y \in \mathcal{F}_x$, we can get

$$d(x_i, y_i) \leq C|x_i|^{1+\gamma} \tag{4.3}$$

and therefore by applying Lemma 3.3 we get

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|} \leq I_2 \tag{4.4}$$

for some $I_2 > 0$. Also, the construction of ξ' implies (4.1) and therefore (4.2) for any $x, y \in A'$, where $A' \in \xi'$. So by the construction of ξ, we get (1.4) with $I = I_1 I_2^2$ for any $x, y \in A$.

On the other hand, for any $x, y \in B_{\varepsilon_5}(R_0)$ with $y \in \mathcal{F}_x$, we have (4.3). So we can apply Lemma 4.4 to get that inside A, distortion of $|DT|_{E(\mathcal{F})}|$ is bounded. It means that for each $x \in A$, the ratio of the length of $T_1^{-n}(B_\varepsilon(\partial R_0) \cap A)) \cap \mathcal{F}_{x_n}$ and the length of $T_1^{-n}(B_{\varepsilon_0}(\partial R_0) \cap A)) \cap \mathcal{F}_{x_n}$ is uniformly bounded by $\varepsilon/\varepsilon_0$ multiplied by a constant. Notice that the angle between the tangent vectors of $\mathcal{F}$ and the tangent space of $T_1^{-n}(\partial B_\varepsilon(R_0))$ are greater than θ_0. Also notice that by the construction of ξ', for any $A \in \xi$, the size of the set $T_1^{-n}A$ along the fiber direction is much smaller than the size of $T_1^{-n}A'$. Hence, the ratio between $\nu(T_1^{-n}B_\varepsilon(\partial R_0) \cap A))$ and $\nu(T_1^{-n}B_{\varepsilon_0}(\partial R_0) \cap A))$ is bounded by a constant times $\varepsilon/\varepsilon_0 \leq (\varepsilon/\varepsilon_0)^\alpha$ for some $\alpha \in (0, 1]$. Now we use (1.4) to get (1.3).[‡]

[‡]Let us make this argument more precise. We denote with A'_n and $A_n(\varepsilon)$ respectively the backward iterates $T_1^{-n}A'$ of some $A' \in \xi'$ and of the set $A \cap B_\varepsilon(\partial R_0)$ where $A = \cup_{x\in A'}\mathcal{F}_x \cap R_0$.

If $DT_p \neq \mathrm{id}$, then it is a rotation, say S. Hence near p we can write $Tx = Sx + T_r(x)$ where $|T_r(x)| \leq C|x|^{1+\gamma}$. If we write $T^{(i)} = \mathrm{id} + S^{-i} \circ T_r \circ S^{i-1}$, then $T^n = S^n \circ T^{(n)} \circ \cdots \circ T^{(1)}$. It implies that the "width" of the annulus $T_1^{-i} R_0$ is bounded by $C|T_1^{-i}x|^{1+\gamma}$. Then we apply the same arguments to get (1.4) and (1.3). □

Lemma 4.1. *There is a foliation on $\{\mathcal{F}_x\}$ on $TR \setminus \{p\}$ consisting of curves from p to points on $\partial(TR)$ such that for any $x \in TR$, the tangent line of $\mathcal{F}_x$ lies in $\mathcal{C}_x$, and $T\mathcal{F}_x \cap TR = \mathcal{F}_{Tx}$.*

Proof: Denote $E_x = \cap_{n \geq 0} DT^n_{T_1^{-n}x}(\mathcal{C}_{T_1^{-n}x})$ for all $x \in TR \setminus \{p\}$. By Lemma 4.2, we know that sine of the angle between any two vectors in $DT^n_{T_1^{-n}x}(\mathcal{C}_{T_1^{-n}x})$ is less than $(1 - d|x_n|^\gamma) \cdots (1 - d|x_1|^\gamma)$. By (1.5) and Lemma 3.1, the product diverges as $n \to \infty$. So $\{E_x\}$ is a subbundle of the tangent bundle over $TR \setminus \{p\}$. Further, we have $DT_x(E_x) = E_{Tx}$ for all $x \in R$. By Lemma 4.3, we know that $\{E_x\}$

Since the angles between the tangent spaces of the curves $\mathcal{F}_x$ and the tangent spaces of the ϵ-neighborhood of the boundary of R_0 are uniformly bounded away from zero, the length of the curve $\mathcal{F}_x \cap B_\varepsilon(\partial R_0)$, when $x \in A'$, is of order ϵ. Its n-backward iterate in $A_n(\varepsilon)$ will be therefore bounded by a constant times ϵ times $d_{n,M}^{1+\gamma}$, where $d_{n,M}$ is the maximum over the ϵ-compact neighborhood of R_0 of $|T_1^{-i}x|$ (see above; equivalently we set $d_{n,m}$ the minimum of $|T_1^{-i}x|$ over the ϵ-compact neighborhood of R_0). Let us call this upper bound $l_{n,\epsilon}$. We construct then the $l_{n,\epsilon}$-neighborhood of A'_n, $B_{l_{n,\epsilon}}(A'_n)$. Clearly

$$\nu(A_n(\varepsilon)) \leq \nu(B_{l_{n,\epsilon}}(A'_n)) \leq \nu'(A'_{n,\varepsilon}) l_{n,\epsilon}$$

where $A'_{n,\varepsilon} = \{z \in T_1^{-n}A'; d(z, A'_n) \leq l_{n,\epsilon}\}$ and ν' denotes the Riemannian volume on $T_1^{-n}\partial R_0$. Since $A'_{n,\varepsilon}$ is contained in a ball of radius $d_{n,M}^{\frac{1}{1-\theta}} + l_{n,\epsilon}$ and A'_n by construction contains a ball of radius $\frac{d_{n,m}^{\frac{1}{1-\theta}} \sin\theta_0/2}{10m}$, we have that $\nu'(A'_{n,\varepsilon}) \leq \mathrm{const}(d_{n,M}^{\frac{1}{1-\theta}} + l_{n,\epsilon})^{m-1}\gamma_{m-1}$ and $\nu'(A'_n) \geq (\frac{d_{n,m}^{\frac{1}{1-\theta}} \sin\theta_0/2}{10m})^{m-1}\gamma_{m-1}$. But $d_{n,M}, d_{n,m}$ are of order $n^{-\beta}$, with $\beta = 1/\gamma$ (see Lemma 3.1), and since $(1+\gamma)(1-\theta) > 1$, we see immediately that for large n:

$$\nu(A_n(\varepsilon)) \leq C'\nu'(A'_n) l_{n,\epsilon}$$

where C' is a suitable constant, depending on m. Let us now define the following objects: $A_n(\varepsilon_0)$: the backward iterate of $A \cap B_{\varepsilon_0}(\partial R_0)$, l'_{n,ϵ_0}: the minimum length of the backward images of the curves $\mathcal{F}_x \cap B_{\varepsilon_0}(\partial R_0)$, when $x \in A'$; $A'_{n,\varepsilon_0} = \{z \in A'_n; d(z, \partial A'_n) \geq l'_{n,\epsilon_0}\}$ and $B_{l''_{n,\epsilon_0}}(A'_{n,\varepsilon_0})$ the l''_{n,ϵ_0}-neighborhood of A'_{n,ε_0}, being $l''_{n,\epsilon_0} = l'_{n,\epsilon_0}\sin\theta_0$. Moreover by what we already said above and which follows from Lemma 4.3, the bounded distortion property along the points of the backward images of the curves $A \cap B_{\varepsilon_0}(\partial R_0)$, will imply that l'_{n,ϵ_0} will be of the same order as l_{n,ϵ_0} (the maximum length of the backward images of the curves). Taking this into account we get:

$$\nu(A_n(\varepsilon_0)) \geq \nu(B_{l''_{n,\epsilon_0}}(A'_{n,\varepsilon_0})) \geq ((\frac{d_{n,m}^{\frac{1}{1-\theta}} \sin\theta_0/2}{10m} - l''_{n,\epsilon_0})^{m-1}\gamma_{m-1} l''_{n,\epsilon_0}$$

By using as above the uniform bounds on $d_{n,M}, d_{n,m}$ when n is large, we see that $\nu(A_n(\varepsilon_0)) \geq C''\nu'(A'_n) l_{n,\epsilon_0}$, where C'' is a suitable constant depending on m. By dividing $\nu(A_n(\varepsilon))$ and $\nu(A_n(\varepsilon_0))$, we get the desired result.

satisfies the Hölder condition near each x with Hölder constants depending on x. Note that $\{E_x\}$ determines a vector field. We can integrate it to get a family of curves $\{\mathcal{F}_x\}$ from p to boundary points of TR that satisfies $T\mathcal{F}_x \cap TR = \mathcal{F}_{Tx}$. By our assumption, $\{\mathcal{F}_x\}$ is the "strong unstable manifold" at x.

It is easy to see that the curve passing through x is unique, and therefore $\{\mathcal{F}_x\}$ forms a foliation. In fact, if there are two such curves $\mathcal{F}_x$ and $\mathcal{F}'_x$ that pass through x, then we can take a curve Γ close to x joining $y \in \mathcal{F}_x$ and $y' \in \mathcal{F}'_x$ such that the tangent line of Γ is in $\mathcal{C}'$. Let us denote by A_n the area of the "triangle" bounded by the curves $T_1^{-n}\Gamma$, $T_1^{-n}\mathcal{F}_{x,y}$ and $T_1^{-n}\mathcal{F}'_{x,y'}$, and by L_n and L'_n the lengths of the curves $T_1^{-n}\mathcal{F}_{x,y}$ and $T_1^{-n}\mathcal{F}'_{x,y'}$ respectively, where $\mathcal{F}_{x,y}$ is the part of the curve in $\mathcal{F}_x$ between x and y, and $\mathcal{F}'_{x,y'}$ is understood in a similar way. By the assumption stated in part (c), the ratio between A_n and $L_n \cdot L'_n$ tends to infinity, which is a contradiction. $\square$

Lemma 4.2. *For any $v, v' \in \mathcal{C}_x$,*

$$\sin\angle(DT_x(v), DT_x(v')) \leq (1 - d|x|^\gamma)\sin\angle(v, v'),$$

where the symbol $\angle(v, v')$ denotes the angle between the vectors v and v'.

Proof: Note that

$$|\det DT_x|_{E(v,v')}| = \frac{|DT_x(v)| \cdot |DT_x(v')| \cdot \sin\angle(DT_x(v), DT_x(v'))}{|v| \cdot |v'| \cdot \sin\angle(v, v)}$$

and

$$||DT_x|_{E(v)}|| = \frac{|DT_x(v)|}{|v|}, \qquad ||DT_x|_{E(v')}|| = \frac{|DT_x(v)|}{|v'|}.$$

Then the results follows from (1.10). $\square$

Lemma 4.3. *There exist constants $H > 0$, $a > 0$, and $\tau_1 \in (0, 1)$, such that for all $x \in TR\backslash\{p\}$,*

$$d(E_x, E_y) \leq \frac{Hd(x, y)^{\tau_1}}{|x|^{\tau_1}} \qquad \forall y \in B(x, a|x|), \tag{4.5}$$

where $d(E_x, E_y)$ is defined by $d(E_x, E_y) = \sin\angle(v_x, v_y)$, v_x and v_y are the tangent vectors of $\mathcal{F}_x$ and $\mathcal{F}_y$ at x and y respectively chosen in the way that $0 \leq \angle(v_x, v_y) < \pi/2$.

Proof: We note that we only need prove (4.5) for all x in a small neighborhood $\tilde{R} \subset R$ of p, because $DT_x(E_x) = E_{Tx}$, and then the results can be extended to TR.

Take $\tilde{d} \in (0, d)$. Then for each x we can extend $\mathcal{C}_x$ to $\tilde{\mathcal{C}}_x$ such that (1.10) holds with $\tilde{d}$ for all $v \in \mathcal{C}_x$ and $v' \in \tilde{\mathcal{C}}_x$. By (1.5) and the fact that T is $C^{1+\gamma}$, we can write $DT(x) = T_0(x) + T_\gamma(x) + T_h(x)$, where $T_0 = DT_p$, T_γ satisfies $T_\gamma(tx) = t^\gamma T_\gamma(x)$ $\forall t > 0$ and $|T_h(x)| = o(|x|^\gamma)$. So it is easy to see that we can find $\varepsilon_a > 0$ such that $\tilde{\mathcal{C}}_x \cap \mathbb{S}^{m-1}$ contains an ε_a-neighborhood of $\mathcal{C}_x \cap \mathbb{S}^{m-1}$ in $\mathbb{S}^{m-1}$ for all x with $|x|$ small. Moreover, since $\mathcal{C}_x$ is uniformly continuous in (t, ϕ), we can take $a > 0$ and $\tilde{R}$ small such that for all $x \in \tilde{R}$, with $d(x, y) \leq a|x|^\gamma$, $\mathcal{C}_y \subset \tilde{\mathcal{C}}_x$. So if $v \in \mathcal{C}_x$ and $v' \in \mathcal{C}_y$, we have

$$\frac{|\det DT_x|_{E(v,v')}|}{\|DT_x|_{E(v)}\| \cdot \|DT_x|_{E(v')}\|} \leq 1 - \tilde{d}|x|^\gamma.$$

Hence, by the same arguments used in Lemma 4.2 we have

$$\sin \angle(DT_x(v), DT_x(v')) \leq (1 - \tilde{d}|x|^\gamma) \sin \angle(v, v'). \tag{4.6}$$

Take $\tau_1 \in (0, 1)$ such that

$$\Big(1 - \frac{\tilde{d}}{2}|x|^\gamma\Big)\Big(\frac{|Tx|}{|x|} \cdot \frac{d(x, y)}{d(Tx, Ty)}\Big)^{\tau_1} \leq 1 \tag{4.7}$$

for all $x \in \tilde{R}$ close to p with $d(x, y) \leq a|x|$.

Take $0 < a_1 \leq a$ such that if $d(x, y) \leq a_1|x|$, then

$$\|DT(x) - DT(y)\| \leq \bar{C}_2|x|^{\gamma-1} d(x, y)^{\tau_1} \tag{4.8}$$

for some $\bar{C}_2 > 0$. This is possible because of (1.7).

Take $H > 0$ such that $H\tilde{d} > 2\bar{C}_2$.

Let $\mathcal{L} = \{L_x : x \in \tilde{R}\backslash\{p\}\}$ be the set of all line bundles in the tangent bundle over $\tilde{R}$. Clearly DT induces a map $\mathcal{D} : \mathcal{L} \to \mathcal{L}$ given by $(\mathcal{D}\mathcal{L})_x = DT_x(L_{T_1^{-1}x})$, and $\mathcal{E} = \{E_x\}$ is the unique fixed point of $\mathcal{D}$ contained in $\mathcal{C}$. Denote

$$\mathcal{H} = \Big\{\{L_x\} \in \mathcal{L} \cap \mathcal{C} : d(L_x, L_y) \leq \frac{Hd(x, y)^{\tau_1}}{|x|^{\tau_1}} \quad \forall y \in B(x, a_1|x|)\Big\}. \tag{4.9}$$

We show that $\mathcal{D}(\mathcal{H}) \subset \mathcal{H}$. This implies the result since $\{E_x\} = \cap_{n\geq 0}\mathcal{D}^n\mathcal{C}$.

Take $\{L_x\} \in \mathcal{H}$. Let $x, y \in \tilde{R}$ with $d(x, y) \leq a_1|x|$. Take unit vectors $e_x \in L_x$, $e_y \in L_y$. So $\sin \angle(e_x, e_y) \leq H|x|^{-\tau_1} d(x, y)^{\tau_1}$. By (4.6) and (4.8),

$$\begin{aligned}
& \sin \angle(DT_x(e_x), DT_y(e_y)) \\
\leq \;& \sin \angle(DT_x(e_x), DT_x(e_y)) + \sin \angle(DT_x(e_y), DT_y(e_y)) \\
\leq \;& (1 - \tilde{d}|x|^\gamma) \sin \angle(e_x, e_y) + |DT_x(e_y) - DT_y(e_y)| \\
\leq \;& (1 - \tilde{d}|x|^\gamma)\frac{Hd(x, y)^{\tau_1}}{|x|^{\tau_1}} + \bar{C}_2|x|^{\gamma-1} d(x, y)^{\tau_1} \\
= \;& \big[(1 - \tilde{d}|x|^\gamma)H + \bar{C}_2|x|^\gamma\big]\frac{d(Tx, Ty)^{\tau_1}}{|Tx|^{\tau_1}} \cdot \frac{d(x, y)^{\tau_1}}{d(Tx, Ty)^{\tau_1}}\frac{|Tx|^{\tau_1}}{|x|^{\tau_1}}.
\end{aligned}$$

By the choice of H, the quantity in the bracket is less than $1 - \tilde{d}|x|^\gamma/2$. Then by (4.7) the right side of the inequality is less than $H|Tx|^{-\tau_1} d(Tx, Ty)^{\tau_1}$. We get the desired results. □

Lemma 4.4. *There exists $J^* > 0$ such that for any x, y with $d(x_i, y_i) \leq |x_i|^{\bar{\gamma}}$ for some $\bar{\gamma} > 1$, $i = 1, \cdots, n$,*

$$\frac{|DT_1^{-n}(y)|_{E_y(\mathcal{F})}|}{|DT_1^{-n}(x)|_{E_x(\mathcal{F})}|} \leq J^*. \tag{4.10}$$

Proof: Take an integer $\bar{r} \geq 2C_0'/C_0$, where C_0 and C_0' are as in (1.6). We assume that $x_0 \leq 1/(\gamma C_0' k_0)^\beta$ for some $k_0 \geq 1$. Then we take $k_i = (\bar{r}^i - 1)k_0$ for $i = 1, \cdots, \ell - 1$, where $\ell - 1$ is the largest number j such that $k_j < n$. Let $k_\ell = n$. By Lemma 3.1, we know that

$$|x_j|^\gamma \leq 2/(\gamma C_0'(k_0 + j)). \tag{4.11}$$

Hence, (1.6) implies

$$\|DT_{x_{k_i}}^{k_i - k_{i-1}}\| \leq \prod_{j=k_{i-1}}^{k_i - 1} \|DT_{x_j}\| \leq \prod_{j=k_{i-1}}^{k_i - 1} \Big(1 + \frac{2C_1}{\gamma C_0'(k_0 + j)}\Big) \leq \prod_{j=k_{i-1}}^{k_i - 1} \Big(1 + \frac{1}{k_0 + j}\Big)^C$$

for some C larger than $2C/\gamma C_0'$ if k_i is large enough. So the choice of $\bar{r}$ gives

$$\|DT_{x_{k_i}}^{k_i - k_{i-1}}\| \leq \Big(\frac{k_0 + k_i}{k_0 + k_{i-1}}\Big)^C \leq \bar{r}^C \tag{4.12}$$

for all $i \geq 0$.

Let e_x be the unit tangent vector of $\mathcal{F}$ at x. We have

$$\begin{aligned}
\frac{|DT_1^{-n}(y)|_{E_y(\mathcal{F})}|}{|DT_1^{-n}(x)|_{E_x(\mathcal{F})}|} &= \frac{|DT_{x_n}^n(e_{x_n})|}{|DT_{y_n}^n(e_{y_n})|} = \frac{|DT_{x_n}^n(e_{x_n})|}{|DT_{x_n}^n(e_{y_n})|} \cdot \frac{|DT_{x_n}^n(e_{y_n})|}{|DT_{y_n}^n(e_{y_n})|} \\
&= \prod_{i=1}^{\ell} \frac{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{x_{k_i}})|}{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{y_{k_i}})|} \cdot \prod_{j=1}^{n} \frac{|DT_{x_j}(e_{y_j})|}{|DT_{y_j}(e_{y_j})|}.
\end{aligned}$$

By the results of Lemma 4.3 and (4.12), each factor in the first product is bounded by

$$\begin{aligned}
& 1 + \frac{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{x_{k_i}})| - |DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{y_{k_i}})|}{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{y_{k_i}})|} \\
\leq\ & 1 + \frac{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{x_{k_i}} - e_{y_{k_i}})|}{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{y_{k_i}})|} \leq 1 + \frac{\|DT_{x_{k_i}}^{k_i - k_{i-1}}\| \cdot |e_{x_{k_i}} - e_{y_{k_i}}|}{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{y_{k_i}})|} \\
\leq\ & 1 + \frac{\bar{r}^C \cdot BHd(x_{k_i}, y_{k_i})^{\tau_1}}{|x_{k_i}|} \leq 1 + \bar{r}^C BH|x_{k_i}|^{\tau_1(\bar{\gamma} - 1)},
\end{aligned}$$

where we use the fact that $|e_{x_{k_i}} - e_{y_{k_i}}| \leq B \sin\angle(e_{x_{k_i}}, e_{y_{k_i}})$ for some $B > 0$. Also note that by (4.11) and the choice of k_i, $\{|x_{k_i}|\}$ decreases exponentially fast as $i \to \infty$. Since $\bar{\gamma} > 1$, the first product in above equality is convergent.

For the second product, by (1.7) each factor is bounded by

$$1+\frac{|DT_{x_j}(e_{y_j})|-|DT_{y_j}(e_{y_j})|}{|DT_{y_j}(e_{y_j})|}\le 1+\frac{C|x_j|^{\gamma-1}d(x,y)}{|DT_{y_j}(e_{y_j})|}\le 1+\frac{C|x_j|^{\bar\gamma+\gamma-1}}{|DT_{y_j}(e_{y_j})|}.$$

By (4.11) and the fact $\bar\gamma>1$, we know that $\sum_j |x_j|^{\bar\gamma+\gamma-1}$ converges. So the product is also bounded. We get the result. □

5 Proof of Theorem A

In this section we first introduce a subspace V_α of $L^1\equiv L^1(\mathbb{R}^m,\nu)$ with compact unit ball that contains the density function of the invariant measures of the induced map of T with respect to the relatively compact subspace $M\backslash R$. Here we only give a brief description and list some properties we use. We refer to [S] and [K] for more details.

Let f be an $L^1(\mathbb{R}^m,\nu)$ function. If Ω is a Borel subset of $\mathbb{R}^m$, we define the oscillation of f over Ω by the difference of essential supremum and essential infimum of f over Ω:

$$\operatorname{osc}(f,\Omega)=\operatorname*{Esup}_{\Omega} f-\operatorname*{Einf}_{\Omega} f.$$

If $B_\epsilon(x)$ denotes the ball of radius ϵ about the point x, then we get a measurable function $x\to\operatorname{osc}(f,\ B_\epsilon(x))$. The function have the following properties.

Proposition 5.1. *Let $f,f_i,g\in L^\infty(\mathbb{R}^m,\nu)$ with $g\ge 0$, $\varepsilon>0$, and S be a Borel subset of $\mathbb{R}^m$. Then*

(i) $\operatorname{osc}\big(\sum_i f_i,\ B_\varepsilon(\cdot)\big)\le\sum_i\operatorname{osc}\big(f_i,\ B_\varepsilon(\cdot)\big)$,

(ii) $\operatorname{osc}\big(f\chi_S,\ B_\varepsilon(\cdot)\big)\le\operatorname{osc}\big(f,\ S\cap B_\varepsilon(\cdot)\big)\chi_S(\cdot)+2\big[\operatorname*{Esup}_{S\cap B_\varepsilon(\cdot)} f\big]\chi_{B_\varepsilon(S)\cap B_\varepsilon(S^c)}$,

(iii) $\operatorname{osc}\big(fg,\ S\big)\le\operatorname{osc}\big(f,\ S\big)\operatorname*{Esup}_S g+\operatorname{osc}\big(g,\ S\big)\operatorname*{Einf}_S f$.

Proof: See [S] Proposition 3.2. □

Take $0<\alpha<1$ and $\varepsilon_0>0$. We define the α-seminorm of f as

$$|f|_\alpha=\sup_{0<\epsilon\le\epsilon_0}\epsilon^{-\alpha}\int_{\mathbb{R}^m}\operatorname{osc}(f,B_\epsilon(x))d\nu(x). \tag{5.1}$$

We will consider the space of the functions f with bounded α-seminorm, namely,

$$V_\alpha=\left\{f\in L^1:|f|_\alpha<\infty\right\}, \tag{5.2}$$

and equip V_α with the norm

$$\|\cdot\|_\alpha = \|\cdot\|_1 + |\cdot|_\alpha, \tag{5.3}$$

where $\|\cdot\|_1$ denotes the L^1 norm. This space does not depend on the choice of ϵ_0. With the $\|\cdot\|_\alpha$ norm, V_α is a Banach space; moreover according to Theorem 1.13 in [K], the unit ball in V_α is compact in L^1.

Proposition 5.2. *Let $f \in V_\alpha$. Then*

(i) $\|f\|_\infty \le \dfrac{1}{\gamma_m \varepsilon_0^m}\|f\|_\alpha$ *provided* $\varepsilon_0 \le 1$,

(ii) *There exists a ball $B_\varepsilon(x)$ such that* $\operatorname*{Einf}_{B_\varepsilon(x)} f > 0$.

Proof: See [S] Proposition 3.4 and Lemma 3.1. □

To prove Theorem A we need one more ingredient, the so-called Lasota-Yorke's inequality, which will be proved in Section 6. This inequality provides an upper bound on the action of the Perron-Frobenius operator on the elements in V_α. Such an operator will be defined on the subspace $M\backslash R$ with a potential given by the inverse of the determinant of the induced map. We denote it by $\hat{P}f$. We will prove

$$|\hat{P}f|_\alpha \le \eta |f|_\alpha + D\|f\|_1$$

for some $\eta < 1$ and $D < \infty$. This, plus the compactness of the unit ball of V_α in L^1, allow us to invoke the ergodic theorem of Ionescu-Tulcea and Marinescu ([IM], see also [K], Theorem 3.3,) to obtain an invariant probability measure μ absolutely continuous with respect to ν on $M\backslash R$. The measure μ has finite number of ergodic components, and is "unique greatest" in the sense that any other measure absolutely continuous with respect to ν is absolutely continuous with respect to μ.

Proof of Theorem A:

Recall that R is given in Assumption 3. We construct an induced system $(\hat{M}, \hat{T})$. Denote $\hat{M} = M\backslash R$. Let $\hat{T} : \hat{M} \to \hat{M}$ be the first return map of T, so that $\hat{T}(x) = T(x)$ if $x \notin T^{-1}R$, otherwise $\hat{T}(x) = T^{i+1}(x) = T_1^i T_j(x)$ if $x \in T_j^{-1}R$, where i is the smallest positive integer such that $T_1^i T_j(x) \notin R$. We denote $g(x) = |\det DT(x)|^{-1}$, and similarly $\hat{g}(x) = g(x)$ if $x \notin T^{-1}R$ and $\hat{g}(x) = |\det D\hat{T}(x)|^{-1} = |\det DT^{i+1}(x)|^{-1}$ if otherwise. Let $\hat{\nu}$ be the renormalization of the Lebesgue measure ν restricted to $\hat{M}$.

Let P be the Perron-Frobenius operator of T with the potential function $\log g(x)$, i.e.

$$Pf(x) = \sum_{Ty=x} f(y)g(y).$$

Then let $\hat{P}$ be the Perron-Frobenius operator of $\hat{T}$ with the potential function $\log \hat{g}(x)$, i.e.

$$\hat{P}f(x) = \sum_{j=1}^{K} \sum_{i=0}^{\infty} f(T_j^{-1}T_1^{-i}x)\hat{g}(T_j^{-1}T_1^{-i}x). \tag{5.4}$$

By the definition of the induced system, we know that if $x \in M \setminus TR$, then $i = 0$, and if $x \in TR \setminus R$, then $j \neq 1$.

By Proposition 6.2 in the next section we have the Lasota-Yorke's inequality for the induced system $(\hat{M}, \hat{T})$. So $\hat{T}$ has an absolutely continuous invariant probability measure $\hat{\mu}$ on $\hat{M}$ with density function $\hat{h}$ that has finitely many ergodic components.

We extend $\hat{h}$ to $R\backslash\{p\}$ to get a density function h on $M\backslash\{p\}$. That is, if $x \in R\backslash\{p\}$, we let

$$\begin{aligned} h(x) &= \sum_{j=2}^{K} h(T_j^{-1}x)g(T_j^{-1}x) \\ &+ \sum_{j=2}^{K} \sum_{i=1}^{\infty} h(T_j^{-1}T_1^{-i}x)g(T_j^{-1}T_1^{-i}x)g(T_1^{-i}x)\cdots g(T_1^{-1}x). \end{aligned} \tag{5.5}$$

It is clear that h is well defined and nonnegative. Also, by this definition, for $x \in TR\backslash\{p\}$,

$$\begin{aligned} h(T_1^{-1}x) &= \sum_{j=2}^{K} h(T_j^{-1}T_1^{-1}x)g(T_j^{-1}T_1^{-1}x) \\ &+ \sum_{j=2}^{K} \sum_{i=2}^{\infty} h(T_j^{-1}T_1^{-i}x)g(T_j^{-1}T_1^{-i}x)g(T_1^{-i}x)\cdots g(T_1^{-2}x). \end{aligned} \tag{5.6}$$

Note that

$$Ph(x) = h(T_1^{-1}x)g(T_1^{-1}x) + \sum_{j=2}^{K} h(T_j^{-1}x)g(T_j^{-1}x).$$

So if $x \in R\backslash\{p\}$, we substitute $h(T_1^{-1}x)$ in (5.6) and then compare it with (5.5) to get that the right side is equal to $h(x)$. If $x \in TR\backslash R$, we substitute $h(T_1^{-1}x)$ and then compare it with (5.4), using the fact $j \neq 1$ and $\hat{g}(T_j^{-1}T_1^{-i}x) = g(T_j^{-1}T_1^{-i}x)g(T_1^{-i}x)\cdots g(T_1^{-1}x)$, to get $Ph(x) = \hat{P}h(x)$, which is also equal to $\hat{h}(x) = h(x)$. Since outside TR, $\hat{P}f = Pf$ for any f and $\hat{h} = h$, we obtain $Ph = h$ on $M\backslash\{p\}$.

Let μ be the measure on M with density h. Clearly, μ is invariant under T and has the same number of ergodic components as $\hat{\mu}$ does.

Next, we show that μM is finite if $\sum_{i=1}^{\infty} \nu(T_1^{-i}R) < \infty$. Recall $R_0 = TR\backslash R$, and let $R_n = T_1^{-n}R_0$ for $n > 0$. By Remark 1.1, $\operatorname{diam} R_n \to 0$. So we have $R = \sum_{n=1}^{\infty} R_n \cup \{p\}$. Since μ is invariant, we have

$$\mu R_i = \mu R_{i+1} + \sum_{j=2}^{K'} \mu(T_j^{-1}R_i),$$

where we assume that in addition to $T_1^{-1}R \subset U_1$, R has $K'-1$ preimages in $U_2, \cdots, U_{K'}$, where $K' \le K$. Take summation from $i = n$ to infinity, we get

$$\mu R_n = \sum_{j=2}^{K'} \mu\big(T_j^{-1} \bigcup_{i=n}^{\infty} R_i\big) = \sum_{j=2}^{K'} \mu\big(T_j^{-1}T_1^{-n}R\big).$$

Note that $\|\hat{h}\|_\infty \le \infty$ since $\hat{h} \in V_\alpha$, and then note that the Jacobian of T_j^{-1} is less than or equal to 1. We have

$$\mu\big(T_j^{-1}T_1^{-n}R\big) \le \|\hat{h}\|_\infty \nu\big(T_j^{-1}T_1^{-n}R\big) \le \|\hat{h}\|_\infty \nu\big(T_1^{-n}R\big).$$

Hence

$$\mu R = \sum_{n=1}^{\infty} \mu R_n \le \|\hat{h}\|_\infty (K'-1) \sum_{n=1}^{\infty} \nu\big(T_1^{-n}R\big) < \infty. \tag{5.7}$$

Now we prove the last part of the theorem. By Proposition 5.2(ii), there is a ball $B_\varepsilon(z) \subset M \setminus R$ such that $\underset{B_\varepsilon(x)}{\operatorname{Einf}}\, \hat{h} \ge h_* > 0$ for some constant h_*. By the assumption in the statement of Theorem A, there exists $\tilde{N} > 0$ such that $T^{\tilde{N}} B_\varepsilon(z) \supset M$. So for any $x \in M$, there is $y_0 \in B_\varepsilon(z)$ such that $T^{\tilde{N}} y_0 = x$. Since $|\det DT|$ is bounded above, we have $g_* := \inf\{g(y) :\ y \in M\} > 0$. Hence, for every x,

$$h(x) = (P^{\tilde{N}}h)(x) = \sum_{T^{\tilde{N}}y = x} h(y) \prod_{i=0}^{\tilde{N}-1} g(T^i y) \ge h(y_0) \prod_{i=0}^{\tilde{N}-1} g(T^i y_0) \ge h_* g_*^{\tilde{N}}.$$

By splitting R over the disjoint union (mod 0) of the R_n as in (5.7), we get

$$\mu R = \sum_{n=1}^{\infty} \mu R_n \ge (h_* g_*^{\tilde{N}}) g_* \sum_{j=2}^{K'} \nu\big(T_1^{-n}R\big) = \infty.$$

This ends the proof. $\square$

6 A Lasota-Yorke type inequality

Let R be as in Assumption 3. Denote $\hat{T}_{ij} = T_1^i T_j$ and $U_{ij} = \hat{T}_{ij}^{-1}(R_0) = T_j^{-1} R_i$ for $i > 1$ and $U_{0j} = U_j \backslash T_j^{-1} R$. So if $TU_l \not\ni p$, then U_{il} is undefined for any $i > 0$ and $U_{0l} = U_l$. Clearly, $U_{ij} \subset U_j$ for all $i > 0$ and $\{U_{ij}, i \geq 0\}$ are pairwise disjoint.

Lemma 6.1. *There exists $0 < \varepsilon_6 \leq \varepsilon_5$ such that for any $\varepsilon_0 \leq \varepsilon_6$, $\varepsilon \leq \varepsilon_0$ and $\forall x \in \hat{M}$,*

$$2 \sum_{j=1}^{K} \sum_{i=0}^{\infty} \frac{\nu(\hat{T}_{ij}^{-1} B_\varepsilon(\partial R_0) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))} \leq \frac{\lambda \varepsilon^\alpha}{\varepsilon_0^\alpha}, \tag{6.1}$$

where λ is given by Assumption 3(b).

Proof: Note that the sets $\cup_{i=1}^{\infty} \partial U_{ij}$, $j = 1, \cdots, K$, are pairwise separated. So by Assumption 3(b) and the definition of λ in (1.1) we only need prove that there exists $\varepsilon_6 > 0$ such that for any given j, neighborhood if $0 < \varepsilon \leq \varepsilon_0 \leq \varepsilon_6$, then

$$2 \sum_{i=0}^{\infty} \frac{\nu(\hat{T}_{ij}^{-1} B_\varepsilon(\partial R_0) \cap B_{(1-s)\varepsilon_6}(x))}{\nu(B_{(1-s)\varepsilon_6}(x))} \leq \frac{\lambda \varepsilon^\alpha}{\varepsilon_0^\alpha}. \tag{6.2}$$

Take

$$\varepsilon_6 \leq \min\{\varepsilon_5, \varepsilon_3\} \cdot \Big(\frac{\lambda (1-s)^m}{2 C_p I^2} \Big)^{1/\alpha},$$

where ε_3 is given by Assumption 3(c).

Recall that N_s is also given by Assumption 3(c). Reduce ε_6 if necessary such that for any x, the ball $B_{(1-s)\varepsilon_6}(x)$ intersects at most one connected component of the set $\{\hat{T}_{ij}^{-1} B_{\varepsilon_6}(\partial R_0), 0 < i \leq N_s, 1 < j \leq K\}$, which, we remember, are pairwise disjoint. We also require ε_6 small enough such that for any $1 < j \leq K$, $1 \leq i \leq N_s$, the part $\hat{T}_{ij}^{-1} \partial R_0 \cap B_{\varepsilon_6}(x)$ are close to an $(m-1)$ dimensional plane.

Take ε and ε_0 such that $0 < \varepsilon \leq \varepsilon_0 \leq \varepsilon_6$.

We first consider the case $1 \leq i \leq N_s$. Note that $\hat{T}_{ij}^{-1} B_\varepsilon(\partial R_0) \cap B_{(1-s)\varepsilon_0}(x) \subset B_{s\varepsilon}(T_{ij}^{-1} \partial R_0) \cap B_{(1-s)\varepsilon_0}(x)$. The volume of the latter is close to $\gamma_{m-1}((1-s)\varepsilon_0)^{m-1} \cdot 2s\varepsilon = 2s\gamma_{m-1}\varepsilon(1-s)^{m-1}\varepsilon_0^{m-1}$. So $\dfrac{\nu(\hat{T}_{ij}^{-1} B_\varepsilon(\partial R) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))}$ is close to $\dfrac{2s\gamma_{m-1}\varepsilon(1-s)^{m-1}\varepsilon_0^{m-1}}{\gamma_m (1-s)^m \varepsilon_0^m} = \dfrac{2s\gamma_{m-1}\varepsilon}{(1-s)\gamma_m \varepsilon_0}$. Hence, by Assumption 3(b), we know that it is less than $\lambda\varepsilon^\alpha/\varepsilon_0^\alpha$.

Now we consider the case that $i \geq N_s$.

Let $\tilde{\varepsilon} = \varepsilon_0 \Big(\dfrac{2C_p I^2}{\lambda(1-s)^m} \Big)^{1/\alpha}$. We have $\tilde{\varepsilon} \le \varepsilon_5$.

For each i, we take a partition $\xi_i = \{\tilde{A}_{i1}, \tilde{A}_{i2}, \cdots, \}$ satisfying Assumption 4(c) with $n = i$ and $\tilde{\varepsilon} \le \varepsilon_5$. Denote $A_{ik} = \tilde{A}_{ik} \cap B_{\tilde{\varepsilon}}(\partial R_0)$, $A'_{ik} = \tilde{A}_{ik} \cap B_{\varepsilon}(\partial R_0)$, $A_{ijk} = \hat{T}_{ij}^{-1} A_{ik}$ and $A'_{ijk} = \hat{T}_{ij}^{-1} A'_{ik}$. Then we let

$$\mathcal{A} = \{A_{ijk} :\ A'_{ijk} \cap B_{(1-s)\varepsilon_0}(x) \ne \emptyset\}, \qquad \mathcal{A}' = \{A'_{ijk} :\ A_{ijk} \in \mathcal{A}\}.$$

By abusing notations, we may also think that $\mathcal{A}$ and $\mathcal{A}'$ are the unions of the sets they contain.

By the fact

$$\nu A_{ijk} = \int_{A_{ik}} |\det D\hat{T}_{ij}^{-1}(x)| d\nu(x)$$

and Assumption 4(c), we know that

$$\frac{\nu A'_{ijk}}{\nu A_{ijk}} \le \frac{C_p \varepsilon^\alpha}{\tilde{\varepsilon}^\alpha} \cdot I^2 = \frac{C_p I^2 \varepsilon^\alpha \lambda(1-s)^m}{2C_p I^2 \varepsilon_0^\alpha} = \frac{\varepsilon^\alpha \lambda(1-s)^m}{2\varepsilon_0^\alpha}. \tag{6.3}$$

Denote $s^* = \sup\big\{ s(T_1^{-N_s}(z), T_1^{N_s}) :\ z \in B_{\tilde{\varepsilon}}(R_0) \big\}$. Note that by Assumption 4(c), $\operatorname{diam} A_{ik} \le 5m\tilde{\varepsilon} \le 5m\varepsilon_0 \Big(\dfrac{2C_p I^2}{\lambda(1-s)^m} \Big)^{1/\alpha}$. Since $i \ge N_s$, by Assumption 3(c), we have

$$\operatorname{diam} A_{ijk} \le 5m\varepsilon_0 \Big(\frac{2C_p I^2}{\lambda(1-s)^m} \Big)^{1/\alpha} \cdot s^* = s\varepsilon_0. \tag{6.4}$$

So if $A_{ijk} \in \mathcal{A}$, then $A_{ijk} \cap B_{(1-s)\varepsilon_0}(x) \ne \emptyset$, and therefore $A_{ijk} \subset B_{\varepsilon_0}(x)$. That is,

$$\mathcal{A} \subset B_{\varepsilon_0}(x). \tag{6.5}$$

Note that

$$\bigcup_{i=0}^{\infty} \hat{T}_{ij}^{-1} B_\varepsilon(\partial R) \cap B_{(1-s)\varepsilon_0}(x) \subset \mathcal{A}'. \tag{6.6}$$

By (6.3) to (6.6), we get

$$\begin{aligned} & 2\sum_{i=0}^{\infty} \frac{\nu(\hat{T}_{ij}^{-1} B_\varepsilon(\partial R) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))} \le 2 \cdot \frac{\nu\mathcal{A}'}{\nu\mathcal{A}} \cdot \frac{\nu\mathcal{A}}{\nu B_{\varepsilon_0}(x)} \cdot \frac{\mu B_{\varepsilon_0}(x)}{\nu(B_{(1-s)\varepsilon_0}(x)} \\ \le\ & 2 \cdot \frac{\varepsilon^\alpha \lambda(1-s)^m}{2\varepsilon_0^\alpha} \cdot 1 \cdot \frac{\gamma_m \varepsilon_0^m}{\gamma_m (1-s)^m \varepsilon_0^m} = \lambda \frac{\varepsilon^\alpha}{\varepsilon_0^\alpha}. \end{aligned}$$

This is (6.2), the formula we need to show. $\square$

Proposition 6.2. *Assume that $T: M \to M$ satisfies Assumption 1 to 4, and $\hat{T}: \hat{M} \to \hat{M}$ is the reduced system with respect to $\hat{M} = M \backslash R$. Then there exist $\eta < 1$ and $D < \infty$ such that for any $f \in V_\alpha = V_\alpha(\varepsilon_0)$, we have $Pf \in V_\alpha$ and*

$$|\hat{P}f|_\alpha \leq \eta |f|_\alpha + D\|f\|_1$$

for all ε_0 sufficiently small.

Proof: Take $\zeta > 0$ such that for any $\varepsilon \leq \varepsilon_4$,

$$(1 + Js^\alpha\varepsilon^a)(1 + cs^a\varepsilon^\alpha) \leq 1 + \zeta\varepsilon^\alpha, \tag{6.7}$$

where c, J and ε_4 are given in Assumption 4(a) and (b) respectively.

Recall that by Assumption 3(b), $s^\alpha + \lambda \leq \eta_0 < 1$. Take $b > 0$ such that $(s^\alpha + \lambda) + 3K'b\gamma_m^{-1} < 1$, where K' is the number of preimages of p for the map T. Recall also that $\varepsilon_1, \varepsilon_2$, ε_4 and ε_6 are given in Assumption 1(b), 3(b), 4(b) and Lemma 6.1 respectively. Take $\varepsilon_0 \leq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_6\}$ such that

$$\eta := (1 + \zeta\varepsilon_0^\alpha)(s^\alpha + \lambda) + 3K'b\gamma_m^{-1} < 1. \tag{6.8}$$

Denote

$$G_R(x, \varepsilon, \varepsilon_0) = 2\sum_{j=1}^{K}\sum_{i=0}^{N(\varepsilon)} \frac{\nu(\hat{T}_{ij}^{-1}B_\varepsilon(\partial R_0) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))}.$$

Recall that $G_U(x, \varepsilon, \varepsilon_0)$ is given by (1.2) in Assumption 3(b). Note that if ε_0 is small, then $\operatorname{supp} G_U(\cdot, \varepsilon, \varepsilon_0)$ and $\operatorname{supp} G_R(\cdot, \varepsilon, \varepsilon_0)$ are disjoint. Also, by (1.1) and Lemma 6.1, we know that

$$G(\varepsilon, \varepsilon_0) = \sup_{x \in M}\{G_U(x, \varepsilon, \varepsilon_0), G_R(x, \varepsilon, \varepsilon_0)\} \leq \frac{\lambda\varepsilon^\alpha}{\varepsilon_0^\alpha}. \tag{6.9}$$

Then we take

$$D = 2\zeta + 2(1 + \zeta\varepsilon^\alpha)\sup_{\varepsilon \leq \varepsilon_0} G(\varepsilon, \varepsilon_0)\varepsilon^{-\alpha} + K'b\gamma_m^{-1}. \tag{6.10}$$

By (6.9), $G(\varepsilon, \varepsilon_0)\varepsilon^{-\alpha} \leq \lambda\varepsilon_0^{-\alpha}$. We have $D < \infty$.

Let $\varepsilon \leq \varepsilon_0$.

By Proposition 5.1,

$$\operatorname{osc}\big(\hat{P}f,\ B_\varepsilon(x)\big) \leq \sum_{j=1}^{K}\sum_{i=0}^{\infty} \operatorname{osc}\big((f\hat{g}) \circ \hat{T}_{ij}^{-1}\chi_{\hat{T}U_{ij}},\ B_\varepsilon(x)\big)$$

$$\leq \sum_{j=1}^{K}\sum_{i=0}^{\infty}\Big(\operatorname{osc}\big((f\hat{g}) \circ \hat{T}_{ij}^{-1}, B_\varepsilon(x)\big)\chi_{\hat{T}U_{ij}}(x) + \big[2\operatorname*{Esup}_{B_\varepsilon(x)}(f\hat{g}) \circ \hat{T}_{ij}^{-1}\big]\chi_{B_\varepsilon(\partial\hat{T}U_{ij})}(x)\Big)$$

$$=: \sum_{j=1}^{K}\sum_{i=0}^{\infty}\Big(R_{ij}^{(1)}(x)\chi_{\hat{T}U_{ij}}(x) + R_{ij}^{(2)}(x)\chi_{B_\varepsilon(\partial\hat{T}U_{ij})}(x)\Big). \tag{6.11}$$

Denote $y_{ij} = \hat{T}_{ij}^{-1}x$. We can choose $N = N(\varepsilon) > 0$ for each $0 < \varepsilon \le \varepsilon_0$ according to Assumption 4(b).

For $R_{ij}^{(1)}(x)$ with $x \in \hat{T}U_{ij}$, we first consider the case $i \le N(\varepsilon)$. By Assumption 4(a), (b) and (6.7), we have $\hat{g}(y'_{ij})/\hat{g}(y_{ij}) \le (1 + Js^\alpha\varepsilon^\alpha)(1 + cs^\alpha\varepsilon^\alpha) \le 1 + \zeta\varepsilon^\alpha$ if $d(T^{i+1}y_{ij}, T^{i+1}y'_{ij}) \le s\varepsilon$. Hence $\hat{g}(y'_{ij}) \le (1 + \zeta\varepsilon^\alpha)\hat{g}(y_{ij})$ and $\mathrm{osc}\big(\hat{g} \circ \hat{T}_{ij}^{-1},\ B_\varepsilon(x)\big) \le \mathrm{osc}\big(\hat{g},\ B_{s\varepsilon}(y_{ij})) \le 2\zeta\varepsilon^\alpha\hat{g}(y_{ij})$. So we get for almost every x,

$$
\begin{aligned}
&R_{ij}^{(1)}(x) = \mathrm{osc}\big(f\hat{g},\ \hat{T}_{ij}^{-1}B_\varepsilon(x) \cap U_{ij}\big)\\
\le\ & \mathrm{osc}\big(f,\ B_{s\varepsilon}(y_{ij}) \cap U_{ij}\big) \operatorname*{Esup}_{B_{s\varepsilon}(y_{ij})\cap U_{ij}} \hat{g} + \mathrm{osc}\big(\hat{g},\ B_{s\varepsilon}(y_{ij}) \cap U_{ij}\big) \operatorname*{Einf}_{B_{s\varepsilon}(y_{ij})\cap U_{ij}} f\\
\le\ & (1 + \zeta\varepsilon^\alpha)\,\mathrm{osc}\big(f,\ B_{s\varepsilon}(y_{ij}) \cap U_{ij}\big)\hat{g}(y_{ij}) + 2\zeta\varepsilon^\alpha f(y_{ij})\hat{g}(y_{ij}).
\end{aligned}
$$

If $i > N(\varepsilon)$, then we must have $x \in R_0$, and therefore for almost every x,

$$
\begin{aligned}
&R_{ij}^{(1)}(x) = \mathrm{osc}\big(f\hat{g},\ \hat{T}_{ij}^{-1}B_\varepsilon(x) \cap U_{ij}\big)\\
\le\ & \mathrm{osc}\big(f, \hat{T}_{ij}^{-1}B_\varepsilon(x) \cap U_{ij})\big) \operatorname*{Einf}_{\hat{T}_{ij}^{-1}B_\varepsilon(x)\cap U_{ij}} \hat{g} + \mathrm{osc}\big(\hat{g}, \hat{T}_{ij}^{-1}B_\varepsilon(x) \cap U_{ij}\big) \operatorname*{Esup}_{\hat{T}_{ij}^{-1}B_\varepsilon(x)} f\\
\le\ & \mathrm{osc}\big(f,\ \ B_{s\varepsilon}(y_{ij}) \cap U_{ij}\big)\hat{g}(y_{ij}) + \|f\|_\infty \sup_{\hat{T}_{ij}^{-1}B_\varepsilon(x)} \hat{g}.
\end{aligned}
$$

By Assumption 4(b), for any $x \in R_0$, $\displaystyle\sum_{i=N}^{\infty}\big(\sup_{\hat{T}_{ij}^{-1}B_\varepsilon(x)} \hat{g}) \le b\varepsilon^{m+\alpha}$. Hence,

$$
\begin{aligned}
&\sum_{j=1}^{K}\sum_{i=0}^{\infty} R_{ij}^{(1)}(x)\chi_{\hat{T}U_{ij}}(x) \le K'b\varepsilon^{m+\alpha}\|f\|_\infty\chi_{R_0}(x)\\
+\ & \sum_{j=1}^{K}\sum_{i=0}^{\infty}\Big((1 + \zeta\varepsilon^\alpha)\,\mathrm{osc}\big(f,\ B_{s\varepsilon}(y_{ij}) \cap U_{ij}\big)\hat{g}(y_{ij}) + 2\zeta\varepsilon^\alpha f(y_{ij})\hat{g}(y_{ij})\Big)\\
\le\ & K'b\varepsilon^{m+\alpha}\|f\|_\infty\chi_{R_0}(x) + (1 + \zeta\varepsilon^\alpha)\big[\hat{P}\,\mathrm{osc}\big(f,\ B_{s\varepsilon}(\cdot)\big)\big](x) + 2\zeta\varepsilon^\alpha(\hat{P}f)(x).
\end{aligned}
$$

Since $\int_{\hat{M}} \hat{P}f d\hat{\nu} = \int_{\hat{M}} f d\hat{\nu}$ for any integrable function f, we have

$$
\begin{aligned}
&\int_{\hat{M}} \sum_{j=1}^{K}\sum_{i=0}^{\infty} R_{ij}^{(1)}\chi_{\hat{T}U_{ij}} d\hat{\nu}\\
\le\ \ & K'b\varepsilon^{m+\alpha}\|f\|_\infty\hat{\nu}R_0 + (1 + \zeta\varepsilon^\alpha)\int_{\hat{M}} \mathrm{osc}\big(f,\ B_{s\varepsilon}(\cdot)\big)d\hat{\nu} + 2\zeta\varepsilon^\alpha\int_{\hat{M}} f d\hat{\nu}\\
\le\ \ & (1 + \zeta\varepsilon^\alpha)s^\alpha\varepsilon^\alpha|f|_\alpha + 2\zeta\varepsilon^\alpha\|f\|_1 + K'b\varepsilon^{m+\alpha}\|f\|_\infty\hat{\nu}R_0. \qquad (6.12)
\end{aligned}
$$

As for $R_{ij}^{(2)}(x)$, if $i \le N(\varepsilon)$, then we have

$$
\operatorname*{Esup}_{B_\varepsilon(x)}(f\hat{g}) \circ \hat{T}_{ij}^{-1} \le \big[\operatorname*{Esup}_{B_{s\varepsilon}(y_{ij})} |f|\big]\hat{g}(y_{ij})(1 + \zeta\varepsilon^\alpha).
$$

Hence by the same method as in [S], we get that

$$\int_{\hat{M}} \sum_{j=1}^{K} \sum_{i=0}^{N(\varepsilon)} R_{ij}^{(2)} \chi_{B_\varepsilon(\partial \hat{T} U_{ij})} d\hat{\nu} \le 2(1+\zeta\varepsilon^\alpha) G(\varepsilon, \varepsilon_0) \big(\varepsilon_0^\alpha |f|_\alpha + \|f\|_1\big).$$

If $i \ge N(\varepsilon)$, then $\operatorname{Esup}_{B_\varepsilon(x)} (f\hat{g}) \circ \hat{T}_{ij}^{-1} \le \|f\|_\infty \sup_{\hat{T}_{ij}^{-1} B_\varepsilon(x)} \hat{g}$, and

$$\sum_{j=1}^{K} \sum_{i=N(\varepsilon)}^{\infty} R_{ij}^{(2)} \chi_{B_\varepsilon(\partial \hat{T} U_{ij})} \le 2K' \|f\|_\infty \sum_{i=N(\varepsilon)}^{\infty} \sup_{\hat{T}_{ij}^{-1} B_\varepsilon(x)} \hat{g}.$$

Again, by Assumption 4(b) it is bounded by $2K' b \varepsilon^{m+\alpha} \|f\|_\infty$. So we have

$$\begin{aligned} &\int_{\hat{M}} \sum_{j=1}^{K} \sum_{i=0}^{\infty} R_{ij}^{(2)} \chi_{B_\varepsilon(\partial \hat{T} U_{ij})} d\hat{\nu} \\ \le\ & 2(1+\zeta\varepsilon^\alpha) G(\varepsilon, \varepsilon_0) \big(\varepsilon_0^\alpha |f|_\alpha + \|f\|_1\big) + 2K' b \varepsilon^{m+\alpha} \|f\|_\infty \hat{\nu} B_\varepsilon(\partial R_0). \quad (6.13) \end{aligned}$$

We may assume that $\hat{\nu} R_0 + \hat{\nu} B_\varepsilon(\partial R_0) \le 1$. By Proposition 5.2(i) and (5.3) we have that $\varepsilon^{m+\alpha} \|f\|_\infty \le \gamma_m^{-1} \varepsilon^\alpha \|f\|_\alpha$ and $\|f\|_\alpha = |f|_\alpha + \|f\|_1$ respectively. So by (6.11), (6.12) and (6.13), we get

$$\begin{aligned} \int_{\hat{M}} \operatorname{osc}\big(Pf,\ B_\varepsilon(\cdot)\big) d\hat{\nu} \ \le\ & \big[(1+\zeta\varepsilon^\alpha)\big(s^\alpha \varepsilon^\alpha + 2G(\varepsilon, \varepsilon_0)\varepsilon_0^\alpha\big) + 3K' b \gamma_m^{-1} \varepsilon^\alpha\big] |f|_\alpha \\ +\ & \big[2\zeta\varepsilon^\alpha + 2(1+\zeta\varepsilon^\alpha) G(\varepsilon, \varepsilon_0) + 3K' b \gamma_m^{-1} \varepsilon^\alpha\big] \|f\|_1. \end{aligned}$$

Now the result follows by the choice of η and D in (6.8) and (6.10). □

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